Gödel's Completeness Theorem

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Here are notes on Kurt Gödel's 1930 paper, "The completeness of the axioms of the functional calculus of logic," in the translation of Stefan Bauer-Mengelberg, as edited by Jean van Heijenoort in the anthology, *From Frege to Gödel* [2].

I make these notes in preparation for a course of three lectures on the Compactness Theorem to be presented in the school (June 20–24, 2015) of the 5th World Congress and School on Universal Logic in Istanbul (the congress being June 25–30).

Gödel's paper apparently represents the earliest explicit statement of a version of the theorem that is now called the Compactness Theorem. In Gödel's version, the signature is countable ("denumerable").

Gödel defines a proof system and proves its completeness in the sense that every sentence that cannot be disproved ("refuted") has a model. By means of the Compactness Theorem, he derives the result that, in a countable signature, an arbitrary set of sentences has a model, provided the set is consistent (the conjunction of no finite subset can be disproved).

As Gödel explains in his first paragraph, the "functional calculus" of his title is the **restricted functional calculus,** which he defines in a footnote:

> In terminology and symbolism this paper follows *Hilbert and Ackermann 1928.*[1] According to that work, the restricted functional calculus contains the logical expressions that are constructed from propositional variables, X, Y, Z, $\dots$, and functional variables (that is, variables for properties and relations) of type I, $F(x)$, $G(x,y)$, $H(x,y,z)$, $\dots$, by means of the operations $\vee$ (or), $\overline{}$ (not), (x) (for all), (Ex) (there exists), with the variable in the prefixes (x) or (Ex) ranging over individuals *only, not* over functions. A formula of this kind is said to be valid (tautological) if a true proposition results from every substitution of specific propositions and functions for X, Y, Z, $\dots$, and $F(x)$, $G(x,y)$, $\dots$, respectively (for example $(x)[F(x) \vee \overline{F(x)}]$).

Perhaps the expression *type I* is the ancestor of our *first order,* and a *function of type I* takes individuals as its arguments. Such functions are our *relations.* A proposition can then be understood as a nullary relation. I shall translate Gödel's symbolism as follows:

Gödel	I
(x)	$\forall x$
(Ex)	$\exists x$
$\overline{}$	$\neg$
$\&$	$\wedge$
$\sim$	$\leftrightarrow$
$\rightarrow$	$\Rightarrow$

[1] Van Heijenoort spells out the reference in his own bibliography: David Hilbert and Wilhelm Ackermann, *Grundzüge der theoretischen Logik* (Berlin: Springer, 1928).

(I vacillate between using $\Rightarrow$ for implication, because the notion is different from a function's being from one set to another, and using $\to$ because it is easier to write by hand.)

The *restricted* functional calculus is what Church [1] calls the *pure* functional calculus. We just call it first-order logic. But we have made a difference in emphasis. We have a different first-order logic for each *signature.* Gödel just supposes there are infinitely many n-ary functional variables (relation symbols) for each n.

Perhaps for Gödel, a *property* is a singulary relation; a *relation,* a relation of more than one argument. An n-ary relation on a set A determines the function from A^n to 2 that takes the value 1 at a point $\boldsymbol{a}$ of A^n if and only if the relation actually holds at that point.

I shall quote Gödel's ten numbered theorems *verbatim,* with my own commentary interspersed.

Theorem I. *Every valid formula of the restricted functional calculus is provable.*

The proof of this will not be completed until after Theorem VI.

Gödel explains validity in another footnote. In our terminology, $\varphi(\boldsymbol{x})$ is **valid** if $\forall \boldsymbol{x}\ \varphi(\boldsymbol{x})$ is true in every structure; by "well-known theorems" (of Löwenheim and Skolem, presumably), it is enough to be true in every countable structure.

The formula $\varphi(\boldsymbol{x})$ is **satisfiable** if $\exists \boldsymbol{x}\ \varphi(\boldsymbol{x})$ is true in some structure. Thus φ is valid if and only if $\neg\varphi$ is not satisfiable.

Gödel's "system of axioms" is as follows. Again I translate into current symbolism; my convention on parentheses is Shoenfield's [3, pp. 17–8]: $\vee$ is more binding than $\Rightarrow$, and of two occurrences of one of these, the occurrence on the right is the more binding.

The "undefined primitive notions" are $\vee$, $\neg$, and $\forall x$, and, "By means of these, $\wedge$, $\Rightarrow$, $\leftrightarrow$, and $\exists x$ can be defined in a well-known way." The axioms are:

1) $P \vee P \Rightarrow P$,

2) $P \Rightarrow P \vee Q$,

3) $P \vee Q \Rightarrow Q \vee P$,

4) $(P \Rightarrow Q) \Rightarrow R \vee P \Rightarrow R \vee Q$,

5) $\forall x\ F(x) \Rightarrow F(y)$,

6) $\forall x\ (P \vee F(x)) \Rightarrow P \vee \forall x\ F(x)$.

The rules of inference are:

Detachment: from φ and $\varphi \Rightarrow \psi$, infer ψ.

Substitution: "The rule of substitution for propositional and functional variables." Above, P, Q, and R are propositional variables, and $F(x)$ is a functional variable. Church discusses this. There are six axioms above; by substitution we derive infinitely many validities. Alternatively, we could replace the axioms with *axiom schemes,* and forget about the rule of substitution.

Generalization: From $\varphi(x)$, infer $\forall x\ \varphi(x)$.

Change of variables: "Individual variables (free or bound) may be replaced by any others, so long as this does not cause overlapping of the scopes of variables denoted by the same sign."

Gödel points out in a note that Russell and Whitehead do not formulate "all of these" explicitly.

Gödel has noted in the first paragraph that we already have completeness for propositional calculus. Thus, I say, we might as well replace the first four axioms above with one axiom scheme, giving us every tautology of the propositional calculus. Gödel himself will reassert propositional completeness as one of the lemmas below.

We could be more explicit about change of variables:

- $\forall x\ \varphi(x) \Rightarrow \forall y\ \varphi(y)$ is an axiom (y being substitutable for x in φ).

- From $\varphi(x)$, infer $\varphi(y)$, provided x is not a free variable of any hypotheses of a proof.

Gödel states some lemmas without proof:

1. For every tuple $\boldsymbol{x}$ of variables,[2]

$$\forall \boldsymbol{x}\ \varphi(\boldsymbol{x}) \Rightarrow \exists \boldsymbol{x}\ \varphi(\boldsymbol{x}),$$
$$\forall \boldsymbol{x}\ \varphi(\boldsymbol{x}) \wedge \exists \boldsymbol{x}\ \psi(\boldsymbol{x}) \Rightarrow \exists \boldsymbol{x}\ (\varphi(\boldsymbol{x}) \wedge \psi(\boldsymbol{x})),$$
$$\forall \boldsymbol{x}\ \neg\varphi(\boldsymbol{x}) \leftrightarrow \neg\exists \boldsymbol{x}\ \varphi(\boldsymbol{x}).$$

   Note that Gödel has not formally explained the meaning of $\exists x$; perhaps the "well-known" definition alluded to above is that $\exists x\ \varphi$ means $\neg\forall x\ \neg\varphi$.

2. For every n in ω, for every permutation σ of n, for every n-tuple $\boldsymbol{x}$ of variables,
$$\exists \boldsymbol{x}\ \varphi(\boldsymbol{x}) \Rightarrow \exists(\boldsymbol{x} \circ \sigma)\ \varphi(\boldsymbol{x}).$$
   In my notation, $\boldsymbol{x}$ is a function from n to the set of variables.

3. Assuming $\boldsymbol{x}$ is injective as such a function, even if $\boldsymbol{y}$ is not,
$$\forall \boldsymbol{x}\ \varphi(\boldsymbol{x}) \Rightarrow \forall \boldsymbol{y}\ \varphi(\boldsymbol{y}).$$

4. [Blending of quantifiers in a conjunction or disjunction: see the proof of Theorem III. Gödel apparently neglects to say that the variables must be distinct.]

5. For every φ there is φ' in *normal form* such that $\varphi \leftrightarrow \varphi'$. Gödel does not define normal form, but only refers again to Hilbert and Ackermann. Evidently it is as defined by Skolem in his 1920 paper [4]: in our terms, a formula is in **normal form** if it is an **open** (quantifier-free) formula, preceded by some (or no) existential quantifiers, preceded by some (or no) universal quantifiers.

[2] As Gödel explains, he uses lower-case German letters like $\mathfrak{x}$, $\mathfrak{y}$, $\mathfrak{u}$, $\mathfrak{v}$, "and so on," for these tuples.

6. If $\varphi \leftrightarrow \varphi'$, then $\psi \leftrightarrow \psi'$, where φ is a subformula of ψ, and ψ' results from ψ by replacing φ with φ'.

7. Completeness of the first four axioms for propositional calculus. Note that Gödel does not mention any rules of inference at this point.

Theorem I is equivalent to:

THEOREM II. *Every formula of the restricted functional calculus is either refutable or satisfiable* (and moreover satisfiable in the [*sic*] denumerable domain of individuals).

Refutable means having provable negation.

The "class" (set) $\mathfrak{K}$ consists of *sentences* in normal form of the more precise form

$$\forall x \ \dots \ \exists y \ \varphi.$$

That is, the sentence must have at least one universal and one existential quantifier.

THEOREM III. *If every $\mathfrak{K}$-expression is either refutable or satisfiable, so is every expression.*

A note says the restriction to "the denumerable domain of individuals" will be implicit.

Proof. By definition, $\varphi(\boldsymbol{x})$ is satisfiable if and only if $\exists \boldsymbol{x} \ \varphi(\boldsymbol{x})$ is satisfiable.

If $\varphi(\boldsymbol{x})$ is refutable, this means $\neg\varphi(\boldsymbol{x})$ is provable, so $\forall \boldsymbol{x} \ \neg\varphi(\boldsymbol{x})$ is provable by Generalization, so $\exists \boldsymbol{x} \ \varphi(\boldsymbol{x})$ is refutable.

If $\exists \boldsymbol{x} \ \varphi(\boldsymbol{x})$ is refutable, this means $\neg\exists \boldsymbol{x} \ \varphi(\boldsymbol{x})$ is provable, so that $\forall \boldsymbol{x} \ \neg\varphi(\boldsymbol{x})$ is provable, and therefore $\neg\varphi(\boldsymbol{x})$ is provable by Axiom 5, that is, $\varphi(\boldsymbol{x})$ is refutable.

It is now enough to prove the claim for *sentences.*

By Lemma 5, it is enough to prove the claim for *normal sentences.* Let $\mathsf{Q} \ \varphi$ be such. If x and y are distinct variables not occuring in Q, and R is some singulary relation symbol, then the sentence

$$\forall x \ \neg Rx \lor \exists y \ Ry$$

is a tautology and is therefore provable by Lemma 7. But by Lemma 4, the sentence

$$\forall x\, \neg Rx \lor \exists y\, Ry \leftrightarrow \forall x\, \exists y\, (\neg Rx \lor Ry)$$

is provable, by Lemma 4, and therefore so are the following:

$$\begin{gathered} \forall x\, \exists y\, (\neg Rx \lor Ry), \\ \mathsf{Q}\, \varphi \leftrightarrow \mathsf{Q}\, \varphi \land \forall x\, \exists y\, (\neg Rx \lor Ry), \\ \mathsf{Q}\, \varphi \land \forall x\, \exists y\, (\neg Rx \lor Ry) \leftrightarrow \forall x\, \mathsf{Q}\, \exists y\, (\varphi \land (\neg Rx \lor Ry)), \\ \mathsf{Q}\, \varphi \leftrightarrow \forall x\, \mathsf{Q}\, \exists y\, (\varphi \land (\neg Rx \lor Ry)). \end{gathered}$$

The sentence $\forall x\, \mathsf{Q}\, \exists y\, (\varphi \land (\neg Rx \lor Ry))$ being in $\mathfrak{K}$, we are done. □

The **degree** of a sentence in $\mathfrak{K}$ is the number of blocks of universal quantifiers.

THEOREM IV. *If every expression of degree k is either satisfiable or refutable, so is every expression of degree $k+1$.*

Proof. Suppose σ is the sentence $\forall \boldsymbol{x}\, \exists \boldsymbol{y}\, \forall \boldsymbol{u}\, \exists \boldsymbol{v}\, \mathsf{Q}\, \varphi$. Suppose R (taking arguments of the length of $\boldsymbol{xy}$) does not occur in φ. Then

$$(R\boldsymbol{xy} \Rightarrow \forall \boldsymbol{u}\, \exists \boldsymbol{v}\, \mathsf{Q}\, \varphi) \leftrightarrow \forall \boldsymbol{u}\, \exists \boldsymbol{v}\, \mathsf{Q}\, (R\boldsymbol{xy} \Rightarrow \varphi).$$

We make the following abbreviations:

$$\begin{aligned} &\tau_1 \text{ is } \forall \boldsymbol{x}'\, \exists \boldsymbol{y}'\, R\boldsymbol{x}'\boldsymbol{y}' \land \forall \boldsymbol{x}\, \forall \boldsymbol{y}\, (R\boldsymbol{xy} \Rightarrow \forall \boldsymbol{u}\, \exists \boldsymbol{v}\, \mathsf{Q}\, \varphi), \\ &\tau_2 \text{ is } \forall \boldsymbol{x}'\, \exists \boldsymbol{y}'\, R\boldsymbol{x}'\boldsymbol{y}' \land \forall \boldsymbol{x}\, \forall \boldsymbol{y}\, \forall \boldsymbol{u}\, \exists \boldsymbol{v}\, \mathsf{Q}\, (R\boldsymbol{xy} \Rightarrow \varphi), \\ &\tau_3 \text{ is } \forall \boldsymbol{x}'\, \forall \boldsymbol{x}\, \forall \boldsymbol{y}\, \forall \boldsymbol{u}\, \exists \boldsymbol{y}'\, \exists \boldsymbol{v}\, \mathsf{Q}\, (R\boldsymbol{x}'\boldsymbol{y}' \land (R\boldsymbol{xy} \Rightarrow \varphi)). \end{aligned}$$

"Obviously" $\tau_1 \Rightarrow \sigma$. By Lemma 6, $\tau_1 \leftrightarrow \tau_2$; by Lemma 4, $\tau_2 \leftrightarrow \tau_3$. We may assume as an inductive hypothesis that τ_3 is either satisfiable or refutable. If τ_3 is satisfiable, then so is σ. If τ_3 is refutable, then so is τ_1. In particular, taking $R\boldsymbol{xy}$ to be $\forall \boldsymbol{u}\, \exists \boldsymbol{v}\, \mathsf{Q}\, \varphi$, we have

$$\neg(\forall \boldsymbol{x}\, \exists \boldsymbol{y}\, \forall \boldsymbol{u}\, \exists \boldsymbol{v}\, \mathsf{Q}\, \varphi \land \forall \boldsymbol{x}\, \forall \boldsymbol{y}\, (\forall \boldsymbol{u}\, \exists \boldsymbol{v}\, \mathsf{Q}\, \varphi \Rightarrow \forall \boldsymbol{u}\, \exists \boldsymbol{v}\, \mathsf{Q}\, \varphi)).$$

But $\forall \boldsymbol{x}\, \forall \boldsymbol{y}\, (\forall \boldsymbol{u}\, \exists \boldsymbol{v}\, \mathsf{Q}\, \varphi \Rightarrow \forall \boldsymbol{u}\, \exists \boldsymbol{v}\, \mathsf{Q}\, \varphi)$, and therefore $\neg\sigma$. □

"It now remains only to prove"

THEOREM V. *Every formula [*sic*] of degree 1 is either satisfiable or refutable.*

Let $\forall \boldsymbol{x}\ \exists \boldsymbol{y}\ \varphi(\boldsymbol{x}, \boldsymbol{y})$ be such a formula, that is, sentence, where $\boldsymbol{x}$ is an r-tuple, and $\boldsymbol{y}$ is an s-tuple, of distinct variables. All variables come from the sequence $(x_k \colon k \in \omega)$. Let

$$\begin{gathered}
\varphi_1 \text{ be } \varphi(\boldsymbol{x}^1, \boldsymbol{y}^1),\\
\varphi_2 \text{ be } \varphi(\boldsymbol{x}^2, \boldsymbol{y}^2) \land \varphi_1,\\
\dots\dots\dots\dots\dots\dots\\
\varphi_n \text{ be } \varphi(\boldsymbol{x}^n, \boldsymbol{y}^n) \land \varphi_{n-1}.
\end{gathered}$$

Here $k \mapsto \boldsymbol{x}^k$ is the bijection from $\mathbb{N}$ to the set of r-tuples of variables such that, writing $\boldsymbol{x}^k$ as $(x_{k(i)} \colon i < r)$, we have that the map

$$k \mapsto (k(0), \dots, k(r-1), k(0) + \dots + k(r-1))$$

is an order-preserving bijection from $\mathbb{N}$ onto ω^r, where the latter has the right lexicographic ordering. In particular,

$$\boldsymbol{x}^1 = (x_0, \dots, x_0), \quad \boldsymbol{x}^2 = (x_1, x_0, \dots, x_0), \quad \boldsymbol{x}^3 = (x_0, x_1, x_0, \dots, x_0).$$

We also let

$$\boldsymbol{y}^k = (x_{(k-1)s+1}, x_{(k-1)s+2}, \dots, x_{(k-1)s+s}).$$

THEOREM VI. *For every n*

$$[\forall \boldsymbol{x}\ \exists \boldsymbol{y}\ \varphi \Rightarrow \exists x_0 \cdots \exists x_{ns}\ \varphi_n]$$

is provable.

Proof. Use induction. By Lemma 3,

$$\forall \boldsymbol{x}\ \exists \boldsymbol{y}\ \varphi(\boldsymbol{x}, \boldsymbol{y}) \Rightarrow \forall \boldsymbol{x}_1\ \exists \boldsymbol{y}\ \varphi(\boldsymbol{x}_1, \boldsymbol{y}),$$

and by Inference Rule 4,

$$\exists \boldsymbol{y}\ \varphi(\boldsymbol{x}_1, \boldsymbol{y}) \leftrightarrow \exists \boldsymbol{y}_1\ \varphi(\boldsymbol{x}_1, \boldsymbol{y}_1),$$

so

$$\forall \boldsymbol{x}\ \exists \boldsymbol{y}\ \varphi(\boldsymbol{x}, \boldsymbol{y}) \Rightarrow \forall \boldsymbol{x}_1\ \exists \boldsymbol{y}_1\ \varphi(\boldsymbol{x}_1, \boldsymbol{y}_1).$$

By Lemma 1,

$$\forall \boldsymbol{x}_1\ \exists \boldsymbol{y}_1\ \varphi(\boldsymbol{x}_1, \boldsymbol{y}_1) \Rightarrow \exists \boldsymbol{x}_1\ \exists \boldsymbol{y}_1\ \varphi(\boldsymbol{x}_1, \boldsymbol{y}_1).$$

This is just the claim when $n = 1$.

Similarly

$$\forall \boldsymbol{x}\ \exists \boldsymbol{y}\ \varphi(\boldsymbol{x}, \boldsymbol{y}) \Rightarrow \forall \boldsymbol{x}_{n+1}\ \exists \boldsymbol{y}_{n+1}\ \varphi(\boldsymbol{x}_{n+1}, \boldsymbol{y}_{n+1}).$$

Now let

$$\boldsymbol{z}_n = (x_i \colon i \leqslant ns\ \ \&\ \ x_i \text{ is not an entry of } \boldsymbol{x}_{n+1}).$$

By Lemma 2,

$$\exists x_0 \cdots \exists x_{sn}\ \varphi_n \Rightarrow \exists \boldsymbol{x}_{n+1}\ \exists \boldsymbol{z}_n\ \varphi_n.$$

By Lemma 1,

$$\begin{aligned}\forall \boldsymbol{x}_{n+1}\ \exists \boldsymbol{y}_{n+1}\ \varphi(\boldsymbol{x}_{n+1}, \boldsymbol{y}_{n+1}) \wedge \exists \boldsymbol{x}_{n+1}\ \exists \boldsymbol{z}_n\ \varphi_n& \\ \Rightarrow \exists \boldsymbol{x}_{n+1}\ (\exists \boldsymbol{y}_{n+1}\ \varphi(\boldsymbol{x}_{n+1}, \boldsymbol{y}_{n+1}) \wedge \exists \boldsymbol{z}_n\ \varphi_n)&.\end{aligned}$$

Thus

$$\begin{aligned}\forall \boldsymbol{x}\ \exists \boldsymbol{y}\ \varphi(\boldsymbol{x}, \boldsymbol{y}) \wedge \exists x_0 \cdots \exists x_{sn}\ \varphi_n& \\ \Rightarrow \exists \boldsymbol{x}_{n+1}\ (\exists \boldsymbol{y}_{n+1}\ \varphi(\boldsymbol{x}_{n+1}, \boldsymbol{y}_{n+1}) \wedge \exists \boldsymbol{z}_n\ \varphi_n)&.\end{aligned}$$

By Lemmas 4, 6, and 2,

$$\exists \boldsymbol{x}_{n+1}\ (\exists \boldsymbol{y}_{n+1}\ \varphi(\boldsymbol{x}_{n+1}, \boldsymbol{y}_{n+1}) \wedge \exists \boldsymbol{z}_n\ \varphi_n) \Rightarrow \exists x_0 \cdots \exists x_{s(n+1)}\ \varphi_{n+1}.$$

By induction, this establishes the claim for all n. □

Let F_n be the propositional formula that results from replacing the atomic ("elementary") components of φ_n with distinct propositional variables (or nullary relation symbols).

If some F_n is refutable, then so is $\exists x_0 \cdots \exists x_{ns}\ \varphi_n$ (by Rules 2 and 3 and Lemma 1). By the last theorem then, $\forall \boldsymbol{x}\ \exists \boldsymbol{y}\ \varphi(\boldsymbol{x}, \boldsymbol{y})$ is refutable.

Suppose now no F_n is refutable. Then all are satisfiable. Then for all n, there is a structure $\mathfrak{A}_n$ with universe ω in which φ_n is satisfied by $(0, \dots, ns)$. "By familiar arguments" we may assume that, for all relation symbols R appearing in φ, for all $\boldsymbol{a}$ with entries from $\{0, \dots, ns\}$,

$$\mathfrak{A}_n \vDash R\boldsymbol{a} \iff \mathfrak{A}_{n+1} \vDash R\boldsymbol{a}.$$

Now define $\mathfrak{B}$ with universe ω so that

$$\mathfrak{B} \vDash R\boldsymbol{a} \iff \text{for sufficiently large } n,\ \mathfrak{A}_n \vDash R\boldsymbol{a}.$$

"Then it is evident at once that" $\mathfrak{B} \vDash \forall \boldsymbol{x}\ \exists \boldsymbol{y}\ \varphi$. This concludes the proof of Theorem I.

> Let us note that the equivalence now proved, "valid = provable", contains, for the decision problem, a reduction of hte nondenumerable to the denumerable, since "valid" refers to the nondenumerable totality of functions, while "provable" presupposes only the denumerable totality of formal proofs.

We can incorporate identity between individuals. Then we get Theorem VII and Theorem VIII, analogues of Theorems I and II.

Theorem IX. *Every denumerably infinite set of formulas of the restricted functional calculus either is satisfiable (*that is, all formulas of the system are simultaneously satisfiable*) or possesses a finite subsystem whose logical product is refutable.*

"IX follows immediately from" the following, which is our **Compactness Theorem** for countable signatures:

Theorem X. *For a denumerably infinite system of formulas to be satisfiable it is necessary and sufficient that every finite subsystem be satisfiable.*

To prove this, it is enough to look at normal sentences of degree one. Suppose we have a system of sentences

$$\forall \boldsymbol{x}_k \; \exists \boldsymbol{y}_k \; \varphi_k(\boldsymbol{x}_k, \boldsymbol{y}_k),$$

where $\boldsymbol{x}_k$ is an r_k-tuple, and $\boldsymbol{y}_k$ is an s_k-tuple. Obtain a bijection $j \mapsto \boldsymbol{x}^k_j$ as before, each $\boldsymbol{x}^k_j$ being an r_k-tuple. Let $\boldsymbol{y}^k_j$ be an s_k-tuple, and suppose the sequence

$$\boldsymbol{y}^1_1, \boldsymbol{y}^1_2, \boldsymbol{y}^2_1, \boldsymbol{y}^1_3, \boldsymbol{y}^2_2, \boldsymbol{y}^3_1, \boldsymbol{y}^1_4, \dots$$

gives the x_j in order. Now define

$$\psi_1 = \varphi_1(\boldsymbol{x}^1_1, \boldsymbol{y}^1_1),$$
$$\psi_n = \psi_{n-1} \land \varphi_1(\boldsymbol{x}^1_n, \boldsymbol{y}^1_n) \land \dots \land \varphi_n(\boldsymbol{x}^n_1, \boldsymbol{y}^n_1).$$

Then

$$\bigwedge_{1 \leqslant k \leqslant n} \forall \boldsymbol{x}_k \; \exists \boldsymbol{y}_k \; \varphi_k(\boldsymbol{x}_k, \boldsymbol{y}_k) \Rightarrow \exists (\boldsymbol{y}^i_j \colon 2 \leqslant i + j \leqslant n + 1) \; \psi_n.$$

If each ψ_n is satisfiable, then so is the system of all of them. But in this case the original system is satisfiable.

> We can also give a somewhat different turn to Theorem IX...

The final two paragraphs concern independence of the axioms.

References

[1] Alonzo Church. *Introduction to mathematical logic. Vol. I.* Princeton University Press, Princeton, N. J., 1956.

[2] Kurt Gödel. The completeness of the axioms of the functional calculus of logic. In van Heijenoort [5], pages 582–91. First published 1930.

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[4] Thoralf Skolem. Logico-combinatorial investigations in the satisfiability or provability of mathematical propositions: A simplified proof of a theorem by L. Löwenheim and generalizations of the theorem. In van Heijenoort [5], pages 252–63. First published 1920.

[5] Jean van Heijenoort, editor. *From Frege to Gödel: A source book in mathematical logic, 1879–1931.* Harvard University Press, Cambridge, MA, 2002.