Compactness

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In the background will be **Zermelo–Fraenkel set theory,** in the first-order logic of signature $\{\in\}$:

Equality:

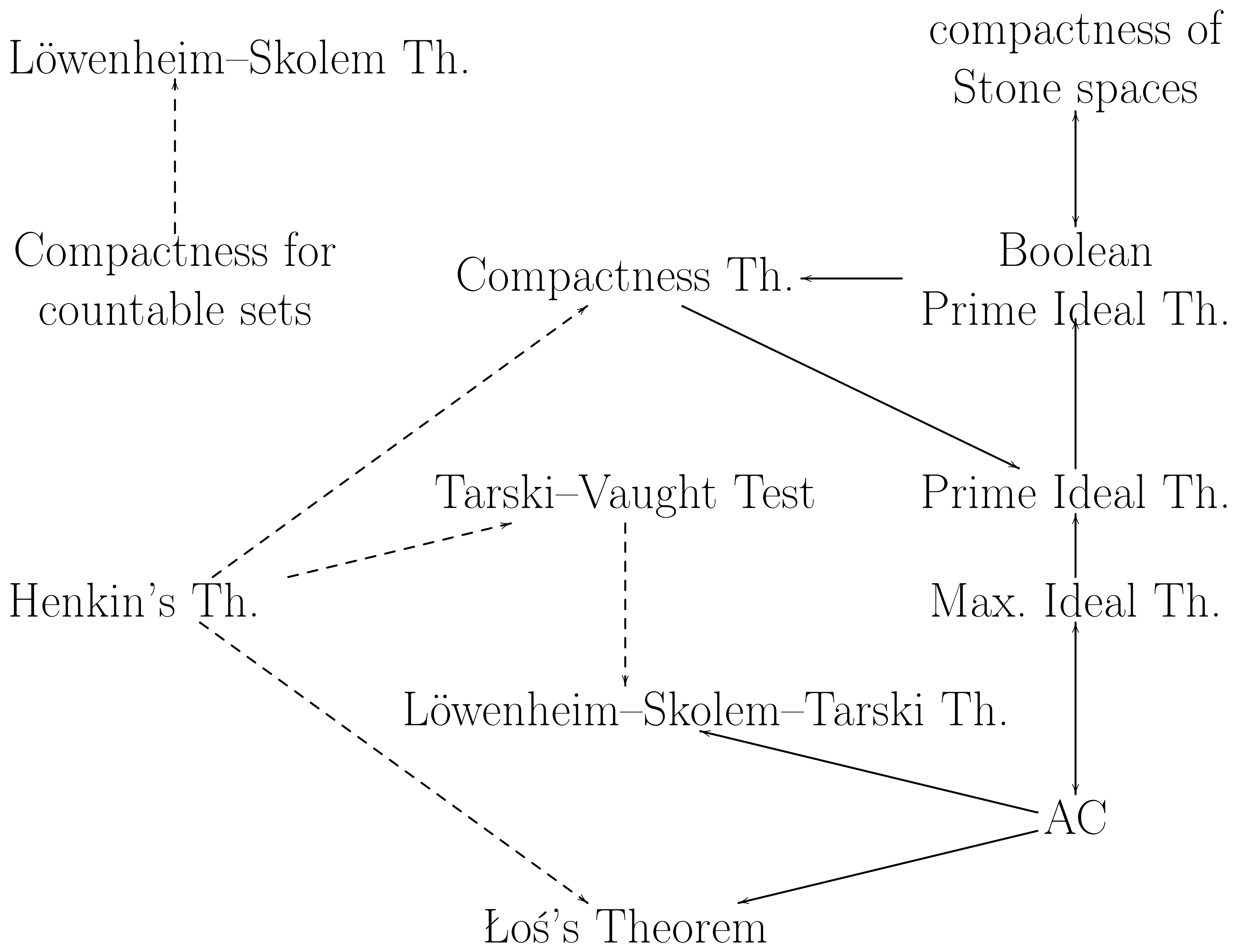
- Equal sets are those having the same elements;
- Equal sets are elements of the same sets.

Comprehension: Every formula $\varphi(x)$ defines the **class**

$$\{x \colon \varphi(x)\}.$$

Certain classes are sets, namely: • the empty class, • a pair of sets, • the union of a set, • the image of a set under a function, • the power class of a set, • $\boldsymbol{\omega}$.

We do not assume the **Axiom of Choice (AC),** which is equivalent to the **Well Ordering Theorem.**


Löwenheim–Skolem Th.
compactness of Stone spaces
Compactness for countable sets
Compactness Th.
Boolean Prime Ideal Th.
Tarski–Vaught Test
Prime Ideal Th.
Henkin's Th.
Max. Ideal Th.
Löwenheim–Skolem–Tarski Th.
AC
Łoś's Theorem

Theorem 1 (1930s?). The **Maximal Ideal Theorem** (for nontrivial, commutative, unital rings) follows from AC.

Proof. A ring $\mathfrak{R}$ with $R = \{a_\xi \colon \xi < \kappa\}$ has maximal ideal

$$\bigcup_{\xi<\kappa} I_\xi, \text{ where } I_\xi = \begin{cases} (a_\xi) + \bigcup_{\eta<\xi} I_\eta, & \text{if this is proper,} \\ \bigcup_{\eta<\xi} I_\eta, & \text{otherwise.} \end{cases}$$

This *is* a proper ideal because the class of commutative $\mathfrak{R}$-algebras without identity is $\forall\exists$-axiomatizable, as by *e.g.*

$$\forall x \, \exists y \; xy \neq y.$$

□

Theorem 2 (Hodges, 1979). The Maximal Ideal Theorem implies the Axiom of Choice.

Theorem 3 (Halpern & Levy, 1971). The Maximal Ideal Theorem does not follow from the **Prime Ideal Theorem.**

The *signature* $\mathscr{S}_{\text{ring}}$ of a ring $\mathfrak{R}$ is $\{0, 1, -, +, \times\}$. Let

$$\operatorname{diag}(\mathfrak{R}) = \{\text{quantifier-free sentences of } \mathscr{S}_{\text{ring}}(R) \text{ true in } \mathfrak{R}\};$$

its models are just the structures in which $\mathfrak{R}$ embeds.

Theorem 4 (Henkin, 1954)**.** The **Prime Ideal Theorem** follows from the **Compactness Theorem** of first-order logic (a *theory* whose every finite subset has a *model* has a model).

Proof. In the signature $\mathscr{S}_{\text{ring}} \cup \{P\}$, let

$$\boldsymbol{K} = \{\text{rings with prime ideal } P\}, \qquad T = \operatorname{Th}(\boldsymbol{K}).$$

Then $\mathbf{Mod}(T) = \boldsymbol{K}$: every model of the theory of $\boldsymbol{K}$ is in $\boldsymbol{K}$.

Every finitely generated sub-ring of a ring $\mathfrak{R}$ has a prime ideal, by Theorem 1.

Hence every finite subset of $T \cup \operatorname{diag}(\mathfrak{R})$ has a model. $\square$

A proper class $\boldsymbol{X}$ can be topologized by

a relation $\models$ ("turnstile") from $\boldsymbol{X}$ to a set B.

If $x \in \boldsymbol{X}$ and $\sigma \in B$ and $x \models \sigma$, say x is a **model** of σ. So we define

$$\mathbf{Mod}(\sigma) = \{x \in \boldsymbol{X} : x \models \sigma\}.$$

If $\Gamma \subseteq B$, we let

$$\mathbf{Mod}(\Gamma) = \bigcap_{\sigma \in \Gamma} \mathbf{Mod}(\sigma).$$

These are the **closed classes** of a **topology** on $\boldsymbol{X}$, assuming (as we do) that for some 0 in B and binary operation $\vee$ on B,

$$\varnothing = \mathbf{Mod}(0), \qquad \mathbf{Mod}(\sigma) \cup \mathbf{Mod}(\tau) = \mathbf{Mod}(\sigma \vee \tau).$$

Call a subset Γ of B **consistent** if

$$\mathbf{Mod}(\Gamma_0) \neq \varnothing, \quad \text{that is,} \quad \bigcap_{\sigma \in \Gamma_0} \mathbf{Mod}(\sigma) \neq \varnothing,$$

for all finite subsets Γ_0 of Γ. If it always follows that $\mathbf{Mod}(\Gamma) \neq \varnothing$, then the topology on $\boldsymbol{X}$ is **compact.**

In any case, we may assume B also has an element 1 and a binary operation $\wedge$ such that

$$\boldsymbol{X} = \mathbf{Mod}(1), \qquad \mathbf{Mod}(\sigma) \cap \mathbf{Mod}(\tau) = \mathbf{Mod}(\sigma \wedge \tau).$$

Now define **logical equivalence** in B by

$$\sigma \sim \tau \iff \mathbf{Mod}(\sigma) = \mathbf{Mod}(\tau).$$

Then $(B, 0, 1, \vee, \wedge)/{\sim}$ is a well-defined distributive lattice.

If $x \in \boldsymbol{X}$, let

$$\mathrm{Th}(x) = \{\sigma \in B \colon x \models \sigma\},$$

$$\begin{tikzcd}
\boldsymbol{X} \arrow[r, squiggly, "\models"] \arrow[d, "x \mapsto \mathrm{Th}(x)"] & B \arrow[d, "\sigma \mapsto \sigma^{\sim}"] \\
\{\mathrm{Th}(x) \colon x \in \boldsymbol{X}\} \arrow[r, squiggly] & B/{\sim}
\end{tikzcd}$$

the **theory** of x. The set of these theories is naturally a **Kolmogorov** (T_0) **quotient** of $\boldsymbol{X}$. Since $0 \notin \mathrm{Th}(x)$ and $1 \in \mathrm{Th}(x)$, while

$$\sigma \vee \tau \in \mathrm{Th}(x) \iff \sigma \in \mathrm{Th}(x) \text{ OR } \tau \in \mathrm{Th}(x),$$

$\mathrm{Th}(x)/{\sim}$ is a **prime filter** of $B/{\sim}$. Let

$$\mathrm{Sto}(B/{\sim}) = \{\text{prime filters of } B/{\sim}\},$$

and if $\sigma \in B$, let $[\sigma] = \{F \in \mathrm{Sto}(B/{\sim}) \colon \sigma^{\sim} \in F\}$. Thus

$$x \in \mathbf{Mod}(\sigma) \iff \mathrm{Th}(x)/{\sim} \in [\sigma].$$

$$
\begin{tikzcd}
\boldsymbol{X} \arrow[r, rightsquigarrow, "\models"] \arrow[d, twoheadrightarrow, "x \mapsto \mathrm{Th}(x)"] & B \arrow[d, twoheadrightarrow, "\sigma \mapsto \sigma^{\sim}"] \\
\{\mathrm{Th}(x) \colon x \in \boldsymbol{X}\} \arrow[r, rightsquigarrow] \arrow[d, rightarrowtail, "\mathrm{Th}(x) \mapsto \mathrm{Th}(x)/\sim"] & B/\sim \arrow[d, rightarrowtail, "\sigma^{\sim} \mapsto [\sigma]"] \\
\mathrm{Sto}(B/\sim) \arrow[r, rightsquigarrow, "\in"] & \{[\sigma] \colon \sigma \in B\}
\end{tikzcd}
$$

Theorem 5.

- If the Prime Ideal Theorem holds, then $\mathrm{Sto}(B/\sim)$ is compact and Kolmogorov (T_0) when topologized by $\{[\sigma] \colon \sigma \in B\}$ under $\in$.
- The map

$$x \mapsto \mathrm{Th}(x)/\sim$$

from $\boldsymbol{X}$ to $\mathrm{Sto}(B/\sim)$ is continuous, and its image is dense and is a Kolmogorov quotient of $\boldsymbol{X}$.

Given a signature $\mathscr{S}$ (such as $\mathscr{S}_{\text{ring}}$), we can let

- $\boldsymbol{X}$ be the class $\mathbf{Str}_{\mathscr{S}}$ of *structures* having signature $\mathscr{S}$,
- B be the set $\mathrm{Sen}_{\mathscr{S}}$ of first-order *sentences* in $\mathscr{S}$,
- $\models$ be the relation of *truth* from $\mathbf{Str}_{\mathscr{S}}$ to $\mathrm{Sen}_{\mathscr{S}}$.

In addition to $\vee$ and $\wedge$, $\mathrm{Sen}_{\mathscr{S}}$ has the operation $\neg$, where

$$\mathbf{Str}_{\mathscr{S}} \smallsetminus \mathbf{Mod}(\sigma) = \mathbf{Mod}(\neg\sigma).$$

Then $\mathrm{Sen}_{\mathscr{S}}/{\sim}$ is a Boolean algebra, called a **Lindenbaum algebra**, so

- its prime filters are *ultrafilters,*
- $\mathrm{Sto}(\mathrm{Sen}_{\mathscr{S}}/{\sim})$ is Hausdorff.

Is the image of $\mathbf{Str}_{\mathscr{S}}$ in $\mathrm{Sto}(\mathrm{Sen}_{\mathscr{S}}/{\sim})$ compact?

A subset Γ of $\mathrm{Sen}_{\mathscr{S}}$ is **complete** if it is consistent and always contains σ or $\neg\sigma$.

Equivalently, $\Gamma = \bigcup \mathscr{U}$ for an ultrafilter $\mathscr{U}$ of $\mathrm{Sen}_{\mathscr{S}}/{\sim}$.

Let $\mathrm{Fm}_{\mathscr{S}}(x) = \{$formulas of $\mathscr{S}$ with free variable $x\}$.

Theorem 6 (Henkin, 1949)**.** Suppose

- T is a complete subset of $\mathrm{Sen}_{\mathscr{S}}$, and
- T has **witnesses:** for every φ in $\mathrm{Fm}_{\mathscr{S}}(x)$, for some constant c in $\mathscr{S}$,

$$T \quad \text{contains} \quad \exists x\, \varphi \to \varphi(c).$$

Then T has a **canonical model,** whose universe consists of its interpretations of the constants in $\mathscr{S}$.

Corollary 6.1 (Mal'cev, 1941)**.** The Prime Ideal Theorem implies the Compactness Theorem.

Proof. Suppose Γ is a consistent subset of $\mathrm{Sen}_{\mathscr{S}}$. We can find

- a set A of constants not in $\mathscr{S}$, together with
- a bijection $\varphi \mapsto c_\varphi$ from $\mathrm{Fm}_{\mathscr{S}(A)}(x)$ to A.

Let $\Gamma^* = \Gamma \cup \{\exists x\, \varphi \to \varphi(c_\varphi) \colon \varphi \in \mathrm{Fm}_{\mathscr{S}(A)}(x)\}$. Then

- Γ^* has witnesses and is consistent;
- the same is true of any complete subset of $\mathrm{Sen}_{\mathscr{S}(A)}$ that includes Γ^*;
- such complete sets exist, by *Lindenbaum's Lemma* (1930, following from the Prime Ideal Theorem). $\square$

Corollary 6.2 (Tarski–Vaught Test, 1957)**.** If $\mathfrak{A} \subseteq \mathfrak{B}$, that is, $\mathfrak{B} \models \operatorname{diag}(\mathfrak{A})$, and if for all φ in $\mathrm{Fm}_{\mathscr{S}}(x)$, for some a in A,

$$\mathfrak{B} \models \exists x\, \varphi \to \varphi(a),$$

then

$$\mathfrak{A} \preccurlyeq \mathfrak{B}$$

($\mathfrak{A}$ is an **elementary substructure** of $\mathfrak{B}$), that is, $\mathfrak{B} \models \operatorname{Th}(\mathfrak{A}_A)$, where $\mathfrak{A}_A$ is the obvious expansion of $\mathfrak{A}$ to $\mathscr{S}(A)$.

Proof. $\mathfrak{A}_A$ is a canonical model of $\operatorname{Th}(\mathfrak{B}_A)$. □

For example,

$$(\{0,1\},+) \nsubseteq (\mathbb{Z},+), \qquad \mathbb{Z} \subseteq \mathbb{Q}, \qquad \mathbb{Z} \not\preccurlyeq \mathbb{Q}, \qquad \mathbb{Q}^{\mathrm{alg}} \preccurlyeq \mathbb{C}.$$

Corollary 6.3 (Löwenheim–Skolem–Tarski Theorem)**.** Every structure of at least the (infinite) cardinality of its signature has an elementary substructure of exactly that cardinality, assuming AC.

Proof. There is a substructure of that size to which the Tarski–Vaught Test applies. □

The **Löwenheim–Skolem Theorem** is the case of countable signatures (this does not need AC).

Hence the **Skolem Paradox:** It is a theorem of ZF that $\mathbb{R}$ is uncountable; but if ZF has a well-ordered model, it has a countable model.

(By Gödel's Second Incompleteness Theorem, it is not a theorem of ZF that ZF has a model at all.)

Corollary 6.4 (Łoś's Theorem, 1955)**.** Assume AC. Suppose

- $(\mathfrak{A}_i : i \in \Omega) \in \mathbf{Str}_{\mathscr{S}}{}^{\Omega}$, and $\mathscr{U}$ is an ultrafilter of $\mathscr{P}(\Omega)$;
- $A = \prod_{i \in \Omega} A_i$, and $\mathfrak{A}_i^*$ is the expansion of $\mathfrak{A}_i$ to $\mathscr{S}(A)$ so that
$$a^{\mathfrak{A}_i^*} = a_i$$
when a is $(a_i : i \in \Omega)$ in A;
- $\|\sigma\| = \{i \in \Omega : \mathfrak{A}_i^* \models \sigma\}$ when $\sigma \in \mathrm{Sen}_{\mathscr{S}(A)}$;
- $T = \big\{\sigma \in \mathrm{Sen}_{\mathscr{S}(A)} : \|\sigma\| \in \mathscr{U}\big\}$ (which is consistent).

Then T has a *canonical* model: an **ultraproduct** of the $\mathfrak{A}_i$.

Proof. If T contains $\exists x\, \varphi$, then it contains $\varphi(a)$, where
$$\mathfrak{A}_i^* \models \exists x\, \varphi \iff \mathfrak{A}_i^* \models \varphi(a_i).$$
$\square$

Theorem 7 (Lindström, 1966)**.** There is no proper "uniform" compact refinement of the topology on $\mathbf{Str}_{\mathscr{S}}$ that retains the Löwenheim–Skolem Theorem.

The Compactness Theorem may be so called because:

1) $\{\mathrm{Th}(\mathfrak{A})/{\sim} : \mathfrak{A} \in \mathbf{Str}_{\mathscr{S}}\}$ is a Stone space, and

2) Stone spaces are compact.

But the work lies in proving, not (2), but (1): In some logics, (1) fails.

Some formulations of the Compactness Theorem are equivalent to the Maximal Ideal Theorem; but it is desirable to recognize the basic form above, equivalent to the Prime Ideal Theorem.

Next June 20–30 in Istanbul: `http://www.uni-log.org/`