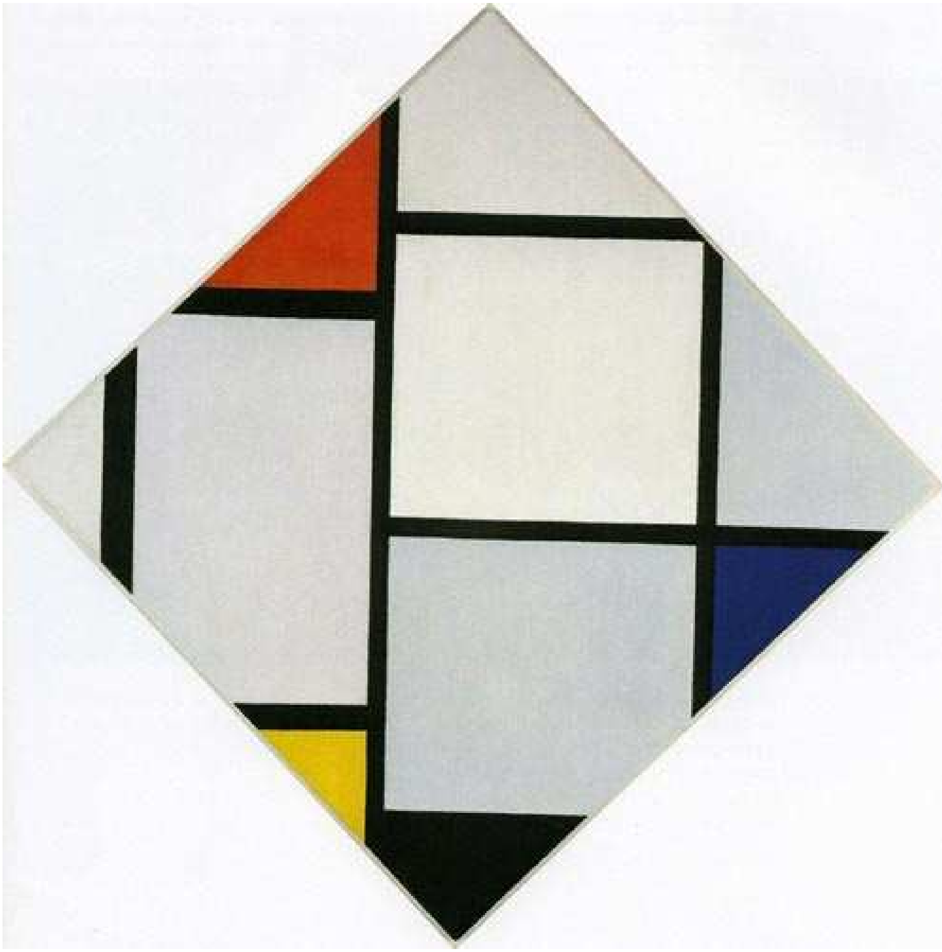

Piet Mondrian, *Tableau No. IV; Lozenge Composition with Red, Gray, Blue, Yellow, and Black*

Interacting Rings

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A **derivation** of a field is an operation D on the field satisfying

$$D(x+y) = Dx + Dy, \qquad D(x \cdot y) = Dx \cdot y + x \cdot Dy.$$

Example. "Taking the derivative,"

$$f \mapsto f',$$

on $\mathbb{R}(x)$ or the field of meromorphic functions on $\mathbb{C}$.

The derivations of a field K compose a **vector space** over K,

$$\operatorname{Der}(K),$$

where the vector-space operations are given by

$$(D_0 + D_1)x = D_0 x + D_1 x, \qquad (a \cdot D)x = a \cdot (Dx).$$

Then $\operatorname{Der}(K)$ also has a **multiplication,** given by

$$[D_0, D_1] = D_0 \circ D_1 - D_1 \circ D_0;$$

this is the **Lie bracket** operation, which I may denote by

$$\mathsf{b}.$$

In this context, a **multiplication** is an operation $\cdot$ on an abelian group that distributes over addition:

$$x \cdot (y+z) = x \cdot y + x \cdot z, \qquad (x+y) \cdot z = x \cdot z + y \cdot z.$$

A **ring** in the most general sense is an abelian group with a multiplication.

Examples.

1. $\mathbb{Z}$ and $\mathbb{Q}$;
2. the **Cayley–Dickson** algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$, $\mathbb{S}$, $\ldots$;
3. the ring $\mathrm{M}_n(R)$ of $n \times n$ matrices over a ring R;
4. $(\mathbb{R}^3, \times)$;
5. $(\mathrm{Der}(K), \mathsf{b})$.

A **group operation** is another kind of multiplication.
The **permutations** of a set A compose a group,

$$(\mathrm{Sym}(A), \circ),$$

the operation being **composition.**
If there is a homomorphism from a group $(G, \cdot)$ to $(\mathrm{Sym}(A), \circ)$,
then $(G, \cdot)$ **acts** on A.
The action is **faithful** if the homomorphism is one-to-one.

Theorem (Cayley)**.** A group acts faithfully on its underlying set.
Indeed, if $(G, \cdot)$ is a group, and $g, x \in G$, define

$$\lambda_g(x) = g \cdot x.$$

Then

$$g \mapsto \lambda_g \colon (G, \cdot) \longrightarrow (\mathrm{Sym}(G), \circ).$$

Now let V be an *abelian* group.
The **endomorphisms** of V compose an abelian group,

$$\mathrm{End}(V).$$

Examples. $\phi \mapsto \phi(1)\colon\ \mathrm{End}(\mathbb{Z}) \xrightarrow{\cong} \mathbb{Z}, \qquad \mathrm{End}(\mathbb{Z} \oplus \mathbb{Z}) \cong \mathrm{M}_2(\mathbb{Z}).$

Then $(\mathrm{End}(V), \circ)$ is an **associative ring:** a ring $(R, \cdot)$ satisfying

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z.$$

If there is a homomorphism from a field K to $(\mathrm{End}(V), \circ)$,
then V is a **vector space** over K.
We may say then K **acts** on V.

Example. K acts on $\mathrm{Der}(K)$.

But also $(\mathrm{Der}(K), \mathsf{b})$ may be said to **act** on K.
So K and $(\mathrm{Der}(K), \mathsf{b})$ are **interacting rings.**

The multiplications of V compose an abelian group,

$$\operatorname{Mult}(V).$$

This has an involutory automorphism, $\mathsf{m} \mapsto \dot{\mathsf{m}}$, where

$$\dot{\mathsf{m}}(x, y) = \mathsf{m}(y, x).$$

Example. $\mathsf{m} \mapsto \mathsf{m}(1, 1)$: $\operatorname{Mult}(\mathbb{Z}) \xrightarrow{\cong} \mathbb{Z}$, but $\dot{\mathsf{m}} = \mathsf{m}$.

Examples. In place of V, consider $\operatorname{End}(V)$:

1. $(\operatorname{End}(V), \circ)$ is an associative ring, as above.
2. $(\operatorname{End}(V), \circ - \dot{\circ})$ is a **Lie ring,** namely, a ring $(R, \cdot)$ in which

   $$(x \cdot y) \cdot z = x \cdot (y \cdot z) - y \cdot (x \cdot z), \qquad x \cdot x = 0.$$

   In particular, $(\operatorname{Der}(K), \mathsf{b})$ is a Lie ring.
3. $(\operatorname{End}(V), \circ + \dot{\circ})$ is a *Jordan ring,* in which

   $$(x \cdot y) \cdot (x \cdot x) = x \cdot (y \cdot (x \cdot x)), \qquad x \cdot y = y \cdot x.$$

If $(R, \cdot)$ is a ring, $p, q \in \mathbb{Z}$, and

$$x \mapsto \lambda_x \colon (R, \cdot) \to (\mathrm{End}(R), p\circ - q\overset{\bullet}{\circ})$$

(where again $\lambda_x(y) = x \cdot y$), let $(R, \cdot)$ be called a **(p, q)-ring.**

Theorem.

1. All associative rings are $(1, 0)$-rings.
2. All Lie rings are $(1, 1)$-rings.

Corollary. If

$$(p, q) \in \{(0, 0), (1, 0), (1, 1)\},$$

then $(\mathrm{End}(V), p\circ - q\overset{\bullet}{\circ})$ is a (p, q)-ring.

Theorem (P)**.** The converse holds.

Proof. $x \mapsto \lambda_x \colon (\mathrm{End}(V), p\circ - q\overset{\bullet}{\circ}) \to (\mathrm{End}(\mathrm{End}(V)), p\circ - q\overset{\bullet}{\circ})$

$$\iff \lambda_{x \cdot y} = \lambda_x \cdot \lambda_y$$

$$\iff \lambda_{px\circ y - qy\circ x}(z) = (p\lambda_x \circ \lambda_y - q\lambda_y \circ \lambda_x)(z)$$

$$\iff p(px \circ y - qy \circ x) \circ z - qz \circ (px \circ y - qy \circ x) = \dots$$

A **differential field** is a pair

$$(K, V),$$

where

1. K is a field,
2. V is both a subspace and a sub-ring of $\operatorname{Der}(K)$.

Theorem. If (K, V) is a differential field, and $\dim_K(V) = m$, then V has a basis

$$\{\partial_0, \dots, \partial_{m-1}\},$$

where in each case

$$[\partial_i, \partial_j] = 0.$$

The *structures* $(K, \partial_0, \dots, \partial_{m-1})$ have a **theory,** which I denote by

$$\mathrm{DF}^m.$$

Example. $\Big(\mathbb{C}(x_0, \dots, x_{m-1}), \partial/\partial x_0, \dots, \partial/\partial x_{m-1}\Big) \models \mathrm{DF}^m$.

Let $\mathfrak{A}$ be a **structure** with underlying set A.
(So $\mathfrak{A}$ might be a group, a differential field, an ordered set, ...)
By introducing *names* for all elements of A, we get the structure

$$\mathfrak{A}_A.$$

The **diagram** of $\mathfrak{A}$ is the quantifier-free theory of $\mathfrak{A}_A$.

Example. The diagram of the field $\mathbb{F}_2$ is axiomatized by

$$\begin{gathered} 0+0=0, \qquad 1+0=1, \qquad 0+1=1, \qquad 1+1=0, \\ 0\cdot 0=0, \qquad 1\cdot 0=0, \qquad 0\cdot 1=0, \qquad 1\cdot 1=1, \\ 0\neq 1. \end{gathered}$$

This does *not* entail field-theory.
For example, it does not entail

$$\forall x\ \forall y\ x\cdot y=y\cdot x.$$

Neither does field-theory entail $1+1=0$.

Let ACF be the theory of **algebraically closed fields,** such as $\mathbb{C}$. That is, ACF has the field axioms, along with, for each positive integer n, the axiom

$$\forall u_0 \ \ldots \forall u_{n-1} \ \exists x \ u_0 + u_1 \cdot x + \cdots + u_{n-1} \cdot x^{n-1} + x^n = 0.$$

Theorem. If K is a field, then the theory

$$\mathrm{ACF} \cup \mathrm{diag}(K)$$

is **complete** (it entails either σ or $\neg\sigma$ for each $\sigma \ldots$).

Proof. Use the *Łoś–Vaught Test.*
(This relies on *Gödel's Completeness Theorem.*)

1. The theory $\mathrm{ACF} \cup \mathrm{diag}(K)$ has no finite models.
2. by Steinitz, all algebraically closed fields that include K, but are of cardinality $(|K| + \aleph_0)^+$, are isomorphic over K. □

(Gödel's *Incompleteness* Theorem: a *particular* theory —namely $\mathrm{Th}(\mathbb{N}, +, \cdot, <)$—has no complete axiomatization.)

Definition (A. Robinson)**.** A theory T is **model complete** if, for all models $\mathfrak{A}$ of T, the theory

$$T \cup \mathrm{diag}(\mathfrak{A})$$

is complete, that is,

$$T \cup \mathrm{diag}(\mathfrak{A}) \vdash \mathrm{Th}(\mathfrak{A}_A).$$

Examples (A. Robinson)**.**

1. Torsion-free divisible abelian groups (*i.e.* vector spaces over $\mathbb{Q}$),
2. algebraically closed fields, such as $\mathbb{C}$ (by the last slide),
3. real-closed fields, such as $\mathbb{R}$.

Theorem (A. Robinson)**.** A theory T is model complete if, for all models $\mathfrak{A}$ of T,

$$T \cup \mathrm{diag}(\mathfrak{A}) \vdash \mathrm{Th}(\mathfrak{A}_A)_\forall,$$

that is, if $\mathfrak{A} \subseteq \mathfrak{B}$, and $\mathfrak{B} \models T$, then:
every **system** over $\mathfrak{A}$ soluble in $\mathfrak{B}$ is soluble in $\mathfrak{A}$.

Let

$$\mathrm{DF}_0^m = \mathrm{DF}^m \cup \{p \neq 0\colon p \text{ prime}\}.$$

Theorem (McGrail, 2000)**.** DF_0^m has a **model companion,** DCF_0^m: that is,

$$(\mathrm{DF}_0^m)_\forall = (\mathrm{DCF}_0^m)_\forall$$

and DCF_0^m is model complete.

Theorem (Yaffe, 2001)**.** The theory of fields of characteristic 0 with m derivations D_i, where

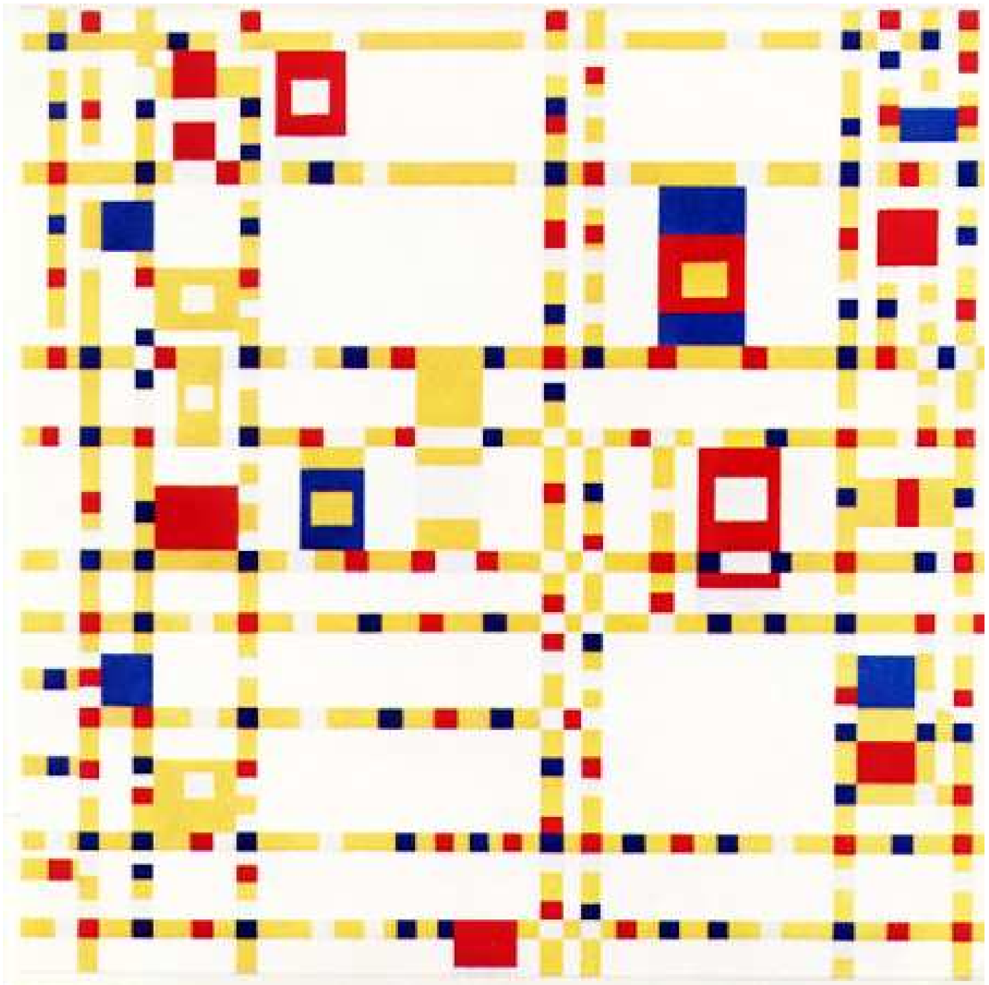
$$[D_i, D_j] = \sum a_{i\,j}^k D_k,$$

has a model companion.

Theorem (P, 2003; Singer, 2007)**.** The latter follows readily from the former.

Theorem (P, submitted March, 2008)**.** DF^m has a model companion, DCF^m, given in terms of varieties.

If (K, V) is a differential field, what is the model theory of V?



Piet Mondrian, *Broadway Boogie Woogie*

Theorem. Let $(V,\cdot)$ be a Lie ring, and

$$R=(\mathrm{End}(V),\circ).$$

Then $(V,\cdot)$ acts on R as a Lie ring of derivations. The action takes D to the derivation

$$f\mapsto Df$$

of R, where

$$Df(x)=D\cdot(f(x))-f(D\cdot x).$$

That is,

$$Df=[\lambda_D,f].$$

In short,

$$D\mapsto\lambda_{\lambda_D}\colon(V,\cdot)\longrightarrow(\mathrm{Der}(R),\mathsf{b}).$$

Again $(V,\cdot)$ is a Lie ring, so it acts on R, namely $(\mathrm{End}(V),\circ)$.
Let $t\in\mathrm{End}(V)$. It may happen that $(\{Dt\colon D\in V\},\circ)$
—is a well-defined sub-ring of R,
—is closed under the action of $(V,\cdot)$, and
—is a field.
Then V is a vector space over K,
and $(V,\cdot)$ acts on K as a ring of derivations.
It may happen further that V acts on K as a *space* of derivations:
That is, if $a,f\in K$ and $D\in V$, it may happen that

$$a(D)f=a\circ(Df).$$

Then let $(V,\cdot,t)$ be called a **vector Lie ring.**

Example. If (K,V) is a differential field, $t\in K$, and $Dt\neq 0$ for some D in V, then (V,b,t) is a vector Lie ring, and

$$(\{Dt\colon D\in V\},\circ)=K.$$

Theorem (P)**.** The class of m-dimensional vector Lie rings is elementary, with $\forall\exists$ axioms. Its theory has a model companion, whose models are those $(V, \cdot, t)$ such that, when we let

$$K = (\{Dt \colon D \in V\}, \circ),$$

then V has a commuting basis $(\partial_i \colon i < m)$ over K, and

$$(K, \partial_0, \dots, \partial_{m-1}) \models \mathrm{DCF}^m .$$

Here $\dim_C(V) = \infty$, where C is the constant field.

However, for an infinite field K, the theory of Lie algebras over K apparently has no model-companion (Macintyre, announced 1973).

Is there a model-complete theory of infinite-dimensional Lie algebras with no extra structure?

Adolph Gottlieb, *Centrifugal*

We can also consider (V, K) as a two-sorted structure.

A vector space can be understood model-theoretically as a triple

$$(V, K, *),$$

where

1. V is an abelian group;
2. K is a field;
3. $*$ is the **action** of K on V, that is,

$$(x, \boldsymbol{v}) \mapsto x * \boldsymbol{v} \colon K \times V \to V,$$

   and $x * \boldsymbol{v} = \lambda_x(\boldsymbol{v})$, where $x \mapsto \lambda_x \colon K \to (\mathrm{End}(V), \circ)$.

Let the theory of vector spaces of dimension n be

$$T_n,$$

where $n \in \{1, 2, 3, \dots, \infty\}$.

Theorem (Kuzichev, 1992)**.** T_n admits elimination of quantified vector-variables.

A theory is **inductive** if unions of chains of models are models.

Theorem (Łoś & Suszko 1957, Chang 1959). A theory T is inductive if and only if

$$T = T_{\forall\exists}.$$

Hence all model complete theories have $\forall\exists$ axioms.

Of an arbitrary T, a model $\mathfrak{A}$ is **existentially closed** if

$$T \cup \mathrm{diag}(\mathfrak{A}) \vdash \mathrm{Th}(\mathfrak{A}_A)_{\forall}.$$

Theorem (Eklof & Sabbagh, 1970). Suppose T is inductive.

1. T has a model companion if and only if the class of its existentially closed models is elementary.

2. In this case, the theory of this class is the model companion.

Again, T_n is the theory of vector spaces of dimension n.
If $n > 1$, then no completion ${T_n}^*$ of T_n can be model complete, because it cannot be $\forall\exists$ axiomatizable.
For example, let

$$a_0 = \boldsymbol{v}^0 = 2, \qquad a_{s+1} = \boldsymbol{v}^{s+1} = \sqrt{a_s},$$
$$K_s = \mathbb{Q}(a_s),$$
$$V_s = \operatorname{span}_{K_s}(\boldsymbol{v}^s, \dots, \boldsymbol{v}^{s+n-1}).$$

Then

$$a_{s+1} * \boldsymbol{v}^{s+1} = \boldsymbol{v}^s,$$

so we have a chain

$$(V_0, K_0) \subseteq (V_1, K_1) \subseteq \cdots$$

of models of T_n whose union has dimension 1.
The situation changes if there are *predicates* for linear dependence.

Let VS_n (where n is a positive integer) be the theory of vector spaces with a new n-ary predicate P^n for linear dependence. So P^n is defined by

$$\exists x^0 \cdots \exists x^{n-1} \left(\sum_{i<n} x^i * \boldsymbol{v}_i = 0 \;\&\; \bigvee_{i<n} x^i \neq 0 \right).$$

Let VS_∞ be the union of the VS_n.

Theorem (P)**.**

1. VS_n has a model companion, the theory of n-dimensional spaces over algebraically closed fields.

2. VS_∞ has a model companion, the theory of infinite-dimensional spaces over algebraically closed fields.

Proof. Given a field-extension L/K, where where

$$[L : K] \geqslant n+1,$$

we can embed (K^{n+1}, K) in (L^n, L), *as models of* VS_n, under

$$\begin{pmatrix} x^0 \\ \vdots \\ x^{n-1} \\ x^n \end{pmatrix} \mapsto \begin{pmatrix} x^0 \\ \vdots \\ x^{n-1} \end{pmatrix} - x^n \begin{pmatrix} a^0 \\ \vdots \\ a^{n-1} \end{pmatrix},$$

where the a^i are chosen from L so that the tuple

$$(a^0, \dots, a^{n-1}, 1)$$

is linearly independent over K. □

Compare:
Let T be the theory of fields with an algebraically closed subfield. The existentially closed models of T have transcendence-degree 1, because of

Theorem (A. Robinson)**.** We have an inclusion

$$K(x, y) \subseteq L(y)$$

of pure transcendental extensions, where

$$K(x, y) \cap L = K,$$

provided

$$L = K(\alpha, \beta),$$

where

$$\alpha \notin K(x, y)^{\mathrm{alg}}, \qquad \beta = \alpha x + y.$$

(Hence T has no model companion.)

A **Lie–Rinehart pair** is a quadruple $(V, K, D, *)$, where

1. V and K are abelian groups,
2. D is an action of V on K; and $*$, of K on V; so

$$(\boldsymbol{u} + \boldsymbol{v})\, D\, x = \boldsymbol{u}\, D\, x + \boldsymbol{v}\, D\, x, \qquad (x + y) * \boldsymbol{v} = x * \boldsymbol{v} + y * \boldsymbol{v},$$

$$\boldsymbol{v}\, D(x + y) = \boldsymbol{v}\, D\, x + \boldsymbol{v}\, D\, y, \qquad x * (\boldsymbol{u} + \boldsymbol{v}) = x * \boldsymbol{u} + x * \boldsymbol{v};$$

3. The actions are faithful:

$$\exists x\ (\boldsymbol{v}\, D\, x = 0 \Rightarrow \boldsymbol{v} = 0), \qquad \exists \boldsymbol{v}\ (x * \boldsymbol{v} = 0 \Rightarrow x = 0);$$

4. if $\boldsymbol{u}, \boldsymbol{v} \in V$, there is a unique element $[\boldsymbol{u}, \boldsymbol{v}]$ of V such that

$$[\boldsymbol{u}, \boldsymbol{v}]\, D\, x = \boldsymbol{u}\, D(\boldsymbol{v}\, D\, x) - \boldsymbol{v}\, D(\boldsymbol{u}\, D\, x),$$

$$(\boldsymbol{u}\, D\, x) * \boldsymbol{v} = [\boldsymbol{u}, x * \boldsymbol{v}] - x * [\boldsymbol{u}, \boldsymbol{v}];$$

5. if $x, y \in K$, there is a unique element $x \cdot y$ of K such that

$$(x \cdot y) * \boldsymbol{v} = x * (y * \boldsymbol{v}),$$

$$(x * \boldsymbol{v})\, D\, y = x \cdot (\boldsymbol{v}\, D\, y).$$

Assuming $(V, K, D, *)$ is a Lie–Rinehart pair, one shows that V does act on K as a Lie ring **of derivations:**

$$\boldsymbol{v}\, D(x \cdot y) = (\boldsymbol{v}\, D\, x) \cdot y + x \cdot (\boldsymbol{v}\, D\, y).$$

Let the theory of those Lie–Rinehart pairs $(V, K, D, *)$ in which $(K, \cdot)$ is a field be denoted by

$$\mathrm{LR}\,.$$

In this case, (K, V) is a differential field.
The theory LR is not inductive. However, let the theory of those models $(V, K, D, *)$ of LR in which

$$\dim_K(V) \leqslant m$$

be denoted by

$$\mathrm{LR}^m\,.$$

Then LR^m is inductive and companionable.

Let T be the theory of pairs (V, K), where K is a field, $\operatorname{char}(K) = 0$, and V acts on K as a space of derivations.
Let $\mathrm{DCF}_0^{(m)}$ be the model-companion of the theory of fields of characteristic 0 with m derivations *with no required interaction.*

Theorem (Özcan Kasal)**.** The existentially closed models of T are just those models (V, K) such that

1. $\operatorname{tr-deg}(K/\mathbb{Q}) = \infty$;
2. $(K, \boldsymbol{v}_0, \dots, \boldsymbol{v}_{m-1}) \models \mathrm{DCF}_0^{(m)}$ whenever $(\boldsymbol{v}_0, \dots, \boldsymbol{v}_{m-1})$ is linearly independent over K;
3. if $(x^0, \dots, x^{n-1})$ is algebraically independent, and $(y^0, \dots, y^{n-1})$ is arbitrary, then for some $\boldsymbol{v}$ in V,
$$\bigwedge_{i<n} \boldsymbol{v}\, D\, x^i = y^i.$$

These are not first-order conditions: they require the constant field to be $\mathbb{Q}^{\mathrm{alg}}$.

The picture changes when (for each n) a predicate Q_n is introduced for the n-ary relation on scalars defined by

$$\bigvee_{i<n} \forall \boldsymbol{v} \ \Bigl(\bigwedge_{j\neq i} \boldsymbol{v}\, D\, x^j = 0 \Rightarrow \boldsymbol{v}\, D\, x^i = 0\Bigr).$$

Let the new theory be

$$T',$$

so this entails

$$\neg Q_n x^0 \cdots x^{n-1} \Leftrightarrow \exists(\boldsymbol{v}_0, \dots, \boldsymbol{v}_{n-1}) \bigwedge_{\substack{i<n \\ j<n}} \boldsymbol{v}_i\, D\, x^j = \delta_i^j.$$

Say $(a^0, \dots, a^{n-1})$ from K is D-**dependent** if

$$(V, K) \models Q_n a^0 \cdots a^{n-1}.$$

So algebraic dependence implies D-dependence.
Also, D-dependence also makes K a pregeometry.

Theorem (Özcan Kasal)**.** The existentially closed models of T' are those (V, K) such that $D\text{-dim}(K) = \infty$ and whenever

1. U is quasi-affine over $\mathbb{Q}(a^0, \dots, a^{k-1}, \vec{b})$ with a generic point
$$(x^0, \dots, x^{\ell+m-1}, \vec{y}),$$
where $\vec{x}$ is algebraically independent over $\mathbb{Q}(\vec{a}, \vec{b})$,
2. $(\boldsymbol{v}_0, \dots, \boldsymbol{v}_{k+\ell-1})$ is linearly independent,
3. $\left(I_k \,\middle|\, 0 \right) = (\boldsymbol{v}_j \, D \, a^i)^{i<k}_{j<k+\ell}$,
4. $\left(\begin{array}{c|c} F & I_\ell \\ \hline G & H \end{array}\right)$ is $(\ell + m) \times (k + \ell)$ with entries from $\mathbb{Q}(\vec{a}, \vec{b})[U]$,

then U contains $(\vec{c}, \vec{d})$ such that

1. $\left(\begin{array}{c|c} F & I_\ell \\ \hline G & H \end{array}\right) (\vec{c}, \vec{d}) = (\boldsymbol{v}_j \, D \, c^i)^{i<\ell+m}_{j<k+\ell}$,
2. $D\text{-dim}(c^\ell, \dots, c^{\ell+m-1}, \vec{d}/\vec{a}, c^0, \dots, c^{\ell-1}) = 0$.

Franz Kline, *Palladio*

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