Differential fields

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This is a transcription, made in August, 2012 (last compiled August 23, 2012), of handwritten notes that I used for a talk given at the mid-term Modnet meeting in Antalya in 2006. The talk was given at the white-board, without slides. I do not know how closely the talk followed these notes. Some sentences or paragraphs of my notes are bracketed in the manuscript, perhaps to indicate that I need not write them on the board. I omit those brackets here. Other parts of the notes are distinguished as being too much to talk about; those parts are omitted here. The abstract of the talk was of course typed up and distributed at the time; it is displayed below. I have now made its defined terms bold, rather than italic. It came with a bibliography, which is now printed as the bibliography of these notes.

In a differential field, how can we tell whether all consistent systems of equations and inequations have solutions? I shall review the history of answers to this question, and I shall update the accounts in [P1, P2].

To begin with the Robinsonian beginnings, I remind or inform the reader-listener of the following. The class of substructures of models of a theory T is elementary, and its theory is $T_\forall$. The class of structures in which a structure $\mathfrak{M}$

embeds is elementary, and its theory is $\mathrm{diag}(\mathfrak{M})$. The class of models of T is closed under unions of chains if and only if $T = T_{\forall\exists}$ [R2, 3.4.7]. The theory T is called **model-complete** [R1] if $T \cup \mathrm{diag}(\mathfrak{M})$ is complete whenever $\mathfrak{M} \models T$. If $T \subseteq T^*$, and $T_\forall = T^*{}_\forall$, then T^* is the **model-completion** [R2] of T if $T^* \cup \mathrm{diag}(\mathfrak{M})$ is complete whenever $\mathfrak{M} \models T$; but T^* is merely the **model-companion** of T if T^* is model-complete. A **derivation** of a field K is an additive endormorphism D of K that respects the Leibniz rule, $D(x \cdot y) = Dx \cdot y + x \cdot Dy$. A **differential field** is a field equipped with one or more derivations.

Various model-complete theories of differential fields are of ongoing interest. It seems worthwhile to review them from the beginning. Some basic definitions are in the abstract.

Example. The theory of the one-dimensional vector-spaces over algebraically closed fields is:

- the **model-completion** of the theory of one-dimensional vector-spaces,
- the **model-companion** of the theory of vector-spaces.

A scalar-field can be made algebraically closed in only one way; but a vector-space of more than one dimension does not determine *how* its dimensions can be collapsed when the scalar-field is enlarged.

I shall talk about:

- DF, the theory of (K, D), where $D \in \mathrm{Der}(K)$;
- DPF, which is $\mathrm{DF} \cup \{\forall x\ \exists y\ (p = 0 \land Dx = 0 \to x^p = y) \colon p \text{ prime}\}$.

Another basic definition:

$\mathfrak{M}$ is an **existentially closed** model of T if

$$\mathfrak{M} \subseteq \mathfrak{N} \models T \implies \mathfrak{M} \preccurlyeq_1 \mathfrak{N},$$

that is, $\mathfrak{M}$ satisfies all quantifier-free formulas with parameters from M that are satisfiable in $\mathfrak{N}$.

A characterization of model-companions:

Theorem (Eklof & Sabbagh, 1971). *If $T = T_{\forall\exists}$, then TFAE:*

- *the class of existentially closed models of T is elementary,*
- *T has a model-companion,*
- *the model-companion of T is the theory of the existentially closed models of T.*

Model-completions meet a stronger condition:

Theorem (Robinson, ⩽1963 [R2]). *TFAE:*

- *T has a model-completion.*
- *$T = T_{\forall\exists}$, and there is $\varphi \mapsto \hat{\varphi}$ on existential (or just primitive) formulas such that, if $\mathfrak{M} \models T$ and $\boldsymbol{a} \in M^n$.*

$$\mathfrak{M} \models \hat{\varphi}(\boldsymbol{a}) \iff \mathfrak{M} \subseteq \mathfrak{N} \models T \cup \{\varphi(\boldsymbol{a})\} \text{ for some } \mathfrak{N},$$

- *the model-[completion] is*

$$T \cup \{\forall \boldsymbol{x}\ (\hat{\varphi}(\boldsymbol{x}) \to \varphi(\boldsymbol{x}))\colon \varphi \text{ existential}\}.$$

The immediate example is the theory of differential fields. Subscripts indicate characteristic; $\mathrm{DF}_0 = \mathrm{DPF}_0$.

Theorem (Seidenberg). *$\varphi \mapsto \hat{\varphi}$ as in Robinson's Theorem exists when T is DF_0 or DPF_p.*

Corollary.

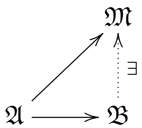
- Robinson, ⩽1963 [R2]: DF_0 *has a model-completion,* DCF_0.
- Wood, 1973 [W1]: DPF_p *has a model-completion,* DCF_p.

For more comprehensible axioms, one can use:

Theorem (Blum, ⩽1977 [B]). *TFAE:*

- *T^* is the model-completion of T,*

- *If* $\mathfrak{A}, \mathfrak{B} \models T$*;* $\mathfrak{M} \models T^*$*;* $\mathfrak{M}$ *is* $|B|^+$*-saturated:*



If, further, $T = T_\forall$*, so substructures of models are models, then the embedding of* $\mathfrak{A}$ *in* $\mathfrak{B}$ *can be analyzed as*

$$\mathfrak{A} \to \mathfrak{A}(a_1) \to \mathfrak{A}(a_1, a_2) \to \cdots \to \mathfrak{B}$$

where each structure is a model of T*; so* $\mathfrak{B} = \mathfrak{A}(a)$ *suffices* (Blum's Criterion).

Since DF_0 is universal, Blum gets nice axioms for DCF_0. Wood gets similar axioms for DCF_p, but *cannot* use Blum's criterion, since DPF_p is not universal:

Theorem (Blum, ⩽1977 [B], Wood, 1974 [W2]). $(K, D) \models \mathrm{DCF}$ *if and only if:*

- $(K, D) \models \mathrm{DPF}$,
- $K = K^{\mathrm{sep}}$,
- $(K, D) \models \exists x\ (f(x, Dx, \dots, D^{n+1}x) = 0 \wedge g(x, Dx, \dots, D^n x) \neq 0)$ *where* f *and* g *are ordinary polynomials over* K*, and* $g \neq 0$ *and* $\partial_{n+1} f \neq 0$.

Wood makes use of r, where

$$\forall x; (r(x)^p = x \vee (Dx \neq 0 \wedge r(x) = 0).$$

Then DPF_p is universal, so Blum's Criterion can in principle be used. Rather, Wood uses a Primitive Element Theorem of Seidenberg.

Singer (1978 [S]) uses Blum's Criterion to get a model-completion of the theory of *ordered* differential fields. This means altering the condition $K = K^{\mathrm{sep}}$ (and then the last condition). Hrushovski and Itai (2003 [HI]) keep $K = K^{\text{alg}}$ [*sic*], but change the last condition to get many model-complete theories of differential fields.

An alternative approach: First, we could have eliminated inequalities by the usual trick, $x \neq 0 \iff \exists y\; xy = 1$.

Over (K, D), a model of DPF, TFAE:

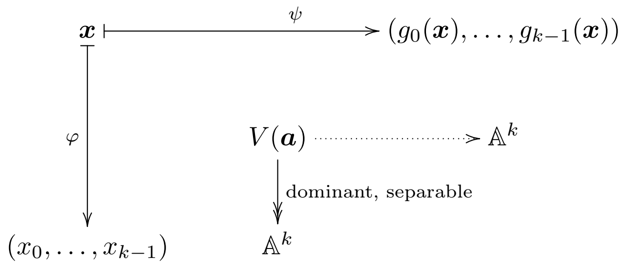
$$\exists \boldsymbol{x} \bigwedge_f f(\boldsymbol{x}, D\boldsymbol{x}, \dots, D^n\boldsymbol{x}) = 0,$$

$$\exists(\boldsymbol{x}_0, \dots, \boldsymbol{x}_n)\; (\bigwedge_f f(\boldsymbol{x}_0, \dots, \boldsymbol{x}_n) = 0 \wedge \bigwedge_{i<n} D\boldsymbol{x}_i = \boldsymbol{x}_{i+1}).$$

The latter is an instance of

$$\exists(x_0, \dots, x_{n-1})\; (\bigwedge_f f(\boldsymbol{x}) = 0 \wedge \bigwedge_{i<k} Dx_i = g_i(\boldsymbol{x})).$$

If this is witnessed by $\boldsymbol{a}$, WMA $(a_0, \dots, a_{k-1})$ is a separating transcendence basis of $K(\boldsymbol{a})/K$.



DCF says: $V(\boldsymbol{a})$ contains P such that $D(\varphi(P)) = \psi(P)$ (P. & Pillay 1998 [PP]).

How do these ideas work in case of several derivations? DF^m is the theory of $(K, \partial_0, \dots, \partial_{m-1})$, where $\partial_i \in \mathrm{Der}(K)$ and $[\partial_i, \partial_j] = 0$.

Theorem (McGrail, 2000 [McG]). DF_0^m *has a model-completion,* DCF_0^m.

Proof. Use Blum's Criterion. If $\sigma \in \omega^m$, let $\partial^\sigma x$ denote

$$\partial_0{}^{\sigma(0)} \dots \partial_{m-1}{}^{\sigma(m-1)} x.$$

Let $\leqslant$ be the product-order on ω^m:

$$\sigma \leqslant \tau \iff \bigwedge_{i<m} \sigma(i) \leqslant \tau(i);$$
$$|\sigma| = \sum_{i<m} \sigma(i);$$
$$\sigma \lessdot \tau \iff (|\sigma|, \sigma(0), \dots, \sigma(m-1)) < (|\tau|, \tau(0), \dots, \tau(m-1)) \quad \text{lexicographically,}$$
$$K\langle a\rangle = K(\partial^\sigma a \colon \sigma \in \omega^m)$$

If $\partial^\sigma a$ is algebraic over its $\lessdot$-predecessors, then so is $\partial^{\sigma+\tau} a$; (and $\partial^{\sigma+\tau} a \geqslant \partial^\sigma a$). (Picture when $m = 2$.) [There was no picture in my notes.]

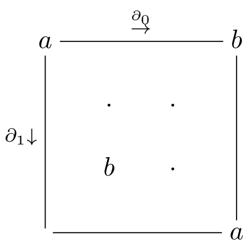
Hence a is a generic zero of a system of finitely many equations. That system can be chosen 'coherent'; being coherent is first-order. □

How can we tell whether an arbitrary system has a solution?

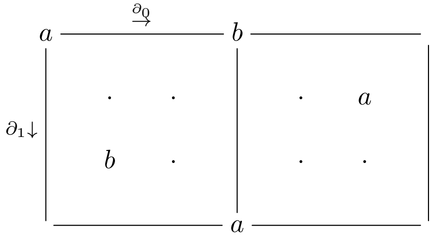
Example. $m = 2$; does

$$\partial^{(n,n)} x = x \wedge \partial^{(n-1,1)} x = \partial^{(0,n)} x$$

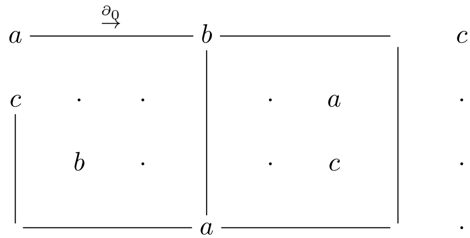
have a solution? $n = 3$:



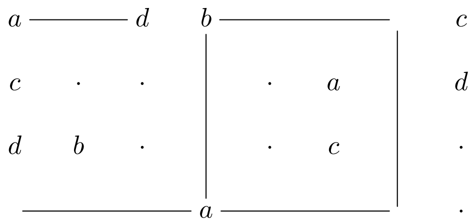
Try differentiating to eliminate $\partial^{(n,n)}x$:



Check the common derivative of b and a:



Check the common derivative of a and c:



A new condition is imposed; what we started with cannot be a solution.

Theorem. *For every m and n, there is M such that, for all models $(K, \partial_0, \dots, \partial_{m-1})$ of DF_0^m, for all fields $K(a^\sigma : \sigma \leqslant (Mn, \dots, Mn))$, if the ∂_i extend so that*

$$\partial_i a^\sigma = a^{\sigma + \boldsymbol{i}}, \qquad (\boldsymbol{i}(j) = \delta_{i\,j}),$$

then $(K, \partial_0, \dots, \partial_{m-1}) \subseteq (L, \tilde{\partial}_0, \dots, \tilde{\partial}_{m-1}) \models \mathrm{DF}_0^m$, where

$$K(a^\sigma : \sigma \leqslant (n, \dots, n)) \subseteq L$$

and $\tilde{\partial}_i a^\sigma = a^{\sigma+\boldsymbol{i}}$.

Here $M \sim m^{m^{\cdot^{\cdot^{\cdot^{m}}}}}$ (a stack of n exponents); but I have some hope that M can be m.

See earlier example.

Differential forms...

References

[R1] Abraham Robinson. *Complete theories.* 1956.

[R2] Abraham Robinson. *Introduction to model theory.* 1963.

[W1] Carol Wood. The model theory of diff. fields of characteristic $p \neq 0$. 1973.

[W2] Carol Wood. Prime model extensions for diff. fields of char. $p \neq 0$. 1974.

[B] Lenore Blum. Differentially closed fields: a model-theoretic tour. 1977.

[S] Michael F. Singer. The model theory of ordered differential fields. 1978.

[PP] D. Pierce and Anand Pillay. A note on the axioms for differentially closed fields of characteristic zero. 1998.

[McG] Tracey McGrail. The model theory of differential fields with finitely many commuting derivations. 2000.

[Y] Yoav Yaffe. Model completion of Lie differential fields. 2001.

[HI] E. Hrushovski and M. Itai. On model complete differential fields. 2003.

[P1] D. Pierce. Diff. forms in the model theory of diff. fields. 2003.

[P2] D. Pierce. Geometric characterizations of e.c. fields with operators. 2004.

[K] Piotr Kowalski. Derivations of the Frobenius map. *J. Symbolic Logic*, 70(1):99–110, 2005.

[T] Marcus Tressl. The uniform companion for large diff. fields of char. 0. 2005.

[MR] Christian Michaux and Cédric Rivière. Quelques remarques concernant la théorie des corps ordonnés différentiellement clos. 2005.

[GR] Nicolas Guzy and Cédric Rivière. Principle of differential lifting for theories of differential fields and Pierce–Pillay axiomatization. 20xx.