REGULAR POLYTOPES

DAVID PIERCE

Contents

1. Polytopes	1
2. Regular polytopes	2
3. Schläfli's criterion	2
4. The regular polyhedra	3
5. The regular polychora	4
5.1. The regular 24-cell	6
5.2. The regular 600-cell	6
References	6

These notes aim to establish the classification of the regular polytopes. The main source is Coxeter's *Regular Polytopes* [2]; also of use are [3] and [1].

1. Polytopes

A polytope is the generalization to arbitrary dimension of a polygon or polyhedron. For a formal definition, let us say that a **polytope** is the *convex hull* of a finite subset of a Euclidean space. Here the **convex hull** of a subset $\{\boldsymbol{v}_k \colon k < m\}$ of $\mathbb{R}^n$ is the set of linear combinations

$$\sum_{k<m} t^k \boldsymbol{v}_k,$$

where $0 \leqslant t^k \leqslant 1$ for each k in m, and $\sum_{k<m} t^k = 1$.

As a vector space, $\mathbb{R}^n$ has the standard basis $(\mathbf{e}_k \colon k < n)$, where

$$\mathbf{e}_k^i = \begin{cases} 1, & \text{if } i = k, \\ 0, & \text{if } i \neq k. \end{cases}$$

Now we can describe three families of polytopes.

1. The convex hull of the $n+1$ points $\mathbf{e}_k$ in $\mathbb{R}^{n+1}$ is an n**-simplex.**
2. The convex hull of the $2n$ points $\pm\mathbf{e}_k$ in $\mathbb{R}^n$ is an n**-orthoplex** (or *cross polytope*).
3. The convex hull of the 2^n points

$$\sum_{k<n} a^k \mathbf{e}_k$$

in $\mathbb{R}^n$, where $a^k = \pm 1$, is an n**-cube.**

Date: March 22, 2010.

We generally identify a polytope with its image under an affine transformation. A polytope is **n-dimensional,** or is an **n-polytope,** if it includes an n-cube. So a 2-polytope is just a polygon; a 3-polytope, a polyhedron. A 4-polytope can be called a **polychoron.**[1]

Suppose K_n is an n-polytope in $\mathbb{R}^n$. A **facet** or **cell** of K_n is the intersection of the polytope with a hyperplane of $\mathbb{R}^n$, provided this intersection is an $(n-1)$-polytope and the original polytope lies on one side of the hyperplane. Suppose we have a sequence $(K_i \colon i < n)$, where each K_i is a facet of K_{i+1}. Then K_2 is a **face;** K_1, an **edge;** and K_0, a **vertex;** of K_n. Suppose $\boldsymbol{v}$ is a vertex of K_{n+1}, and X is the set of midpoints of the edges of K_{n+1} that contain $\boldsymbol{v}$. If the convex hull of X is an n-polytope, then it is a **vertex figure** of K at $\boldsymbol{v}$.

2. Regular polytopes

If p is a positive integer, then the **regular p-gon** is the convex hull in $\mathbb{R}^2$ of the points $(\cos(2\pi k/p), \sin(2\pi k/p))$, where $k < p$. The **regular polytopes** can now be defined recursively:

1. Every 0- and 1-polytope is regular.
2. The regular p-gon is regular for each positive integer p.
3. An $(n+3)$-polytope is regular if
   (a) each of its facets is regular, and
   (b) at each of its vertices, there is a regular vertex figure.

One shows that the facets of a regular polytope are isometric to one another, as are the vertex figures. Then one can denote regular $(n+2)$-polytopes by **Schläfli symbols** as follows. The regular p-gon is denoted by $\{p\}$. A regular $(n+3)$-polytope is denoted by

$$\{p_{n+2}, \dots, p_1\},$$

provided the facets are $\{p_{n+2}, \dots, p_2\}$ and the vertex figures are $\{p_{n+1}, \dots, p_1\}$. Then the faces of $\{p, \dots\}$ are p-gons. An $(n+2)$-simplex is $\{3, \dots, 3\}$; an $(n+3)$-orthoplex, $\{3, \dots, 3, 4\}$; an $(n+3)$-cube, $\{4, 3, \dots, 3\}$.

3. Schläfli's criterion

Let K_{n+1} be an $(n+1)$-gon $\{p_n, \dots, p_1\}$. We want to know the possibilities for the p_m. To this end, we consider successive vertex figures. If $m \leqslant n$, let K_m denote $\{p_{m-1}, \dots, p_1\}$. Then K_m is the vertex figure of K_{m+1}. In K_{m+1}, we consider three parameters:

(1) the segment R_m from a vertex to the center;
(2) the segment ℓ_m from a vertex to the center of an adjacent edge;
(3) the angle ϕ_m at the center of K_{m+1} subtended by ℓ_m.

We have first of all

$$\ell_m = R_m \sin \phi_m.$$

[1]Wikipedia (accessed March 11, 2010) attributes the name *polychoron* to Norman Johnson (a student of Coxeter) and George Olshevski. The Greek source for the latter half of the name is apparently χῶρος 'piece of ground' rather than χορός 'dance, chorus'.

Two adjacent edges of K_{m+1} are adjacent edges of a p_m-gon; moreover, the midpoints of these edges are the endpoints of an edge of K_m; this yields the relation

$$\ell_{m-1} = \ell_m \cos\frac{\pi}{p_m}.$$

Consequently

$$R_{m-1}\sin\phi_{m-1} = \ell_m \cos\frac{\pi}{p_m}.$$

Finally, the center of K_m lies on the corresponding R_m; in particular, the center of K_m is at the foot of a perpendicular to R_m dropped from a vertex of K_m. But this vertex is the midpoint of an edge of K_{m+1} adjacent to R_m. Therefore

$$R_{m-1} = \ell_m \cos\phi_m.$$

Now we obtain

$$\cos\phi_m = \frac{\cos(\pi/p_m)}{\sin\phi_{m-1}}$$

and therefore

$$\sin^2\phi_m = 1 - \frac{\cos^2(\pi/p_m)}{\sin^2\phi_{m-1}}.$$

If we write

$$\sin^2\phi_{m-1} = \frac{\Delta_m}{\Delta_{m-1}}, \tag{$*$}$$

then we have

$$\sin^2\phi_m = \frac{\Delta_m - \Delta_{m-1}\cos^2(\pi/p_m)}{\Delta_m}.$$

Since $\phi_1 = \pi/p_1$, we obtain $(*)$ when we recursively define

$$\Delta_1 = 1, \qquad \Delta_2 = \sin^2\frac{\pi}{p_1}, \qquad \Delta_{m+2} = \Delta_{m+1} - \Delta_m\cos^2\frac{\pi}{p_{m+1}}.$$

Consequently,

$$\Delta_{n+1} = \sin^2\phi_n \cdots \sin^2\phi_1,$$

so we have **Schläfli's criterion,** a necessary condition for the existence of K_{n+1}:

$$\Delta_{n+1} > 0.$$

4. The regular polyhedra

We compute in particular

$$\Delta_3 = \sin^2\frac{\pi}{p_1} - \cos^2\frac{\pi}{p_2},$$

which yields the condition $\sin(\pi/p_1) > \cos(\pi/p_2)$; since $\cos\alpha = \sin(\pi/2-\alpha)$, the condition can be written as

$$\frac{1}{2} < \frac{1}{p_1} + \frac{1}{p_2}.$$

Hence the Schläfli symbol of a regular polyhedron must be one of

$$\{3,3\}, \qquad \{3,4\}, \qquad \{4,3\}, \qquad \{3,5\}, \qquad \{5,3\}.$$

Moreover, such polyhedra do exist. Indeed, the first three are the 3-simplex (that is, the tetrahedron), the 3-orthoplex (the octahedron), and the 3-cube (the cube). The last two are the dodecahedron and the icosahedron, which can be defined as follows. Let ϕ be the **golden ratio,** determined by two conditions:

$$\phi > 0, \qquad \phi = \frac{1}{\phi - 1}.$$

Then also

$$\phi - 1 = \phi^{-1}, \qquad \phi^2 - \phi = 1, \qquad 1 - \phi^{-1} = \phi^{-2}.$$

Consequently the points (ϕ, ϕ^{-1}), $(1, 1)$, and $(0, \phi)$ in $\mathbb{R}^2$ are collinear. So the points

$$(\phi, \phi^{-1}, 0), \qquad (1, 1, 1), \qquad (0, \phi, \phi^{-1}), \qquad (0, \phi, -\phi^{-1}), \qquad (1, 1, -1)$$

in $\mathbb{R}^3$ are coplanar. Indeed, these five points, in the given order, are the vertices of a regular pentagon. For, call the points A, B, C, D, and E. Then the square of the length of AB, BC, DE, or EA is in each case given by the computation

$$(\phi - 1)^2 + (\phi^{-1} - 1)^2 + 1 = \phi^{-2} + (\phi^{-1} - 1)^2 + 1 = 2(\phi^{-2} - \phi^{-1} + 1) = 4\phi^{-2};$$

and $2\phi^{-1}$ is the length of CD. Also the square of the length of AC, BD, CE, or DA is in each case given by

$$\phi^2 + (\phi - \phi^{-1})^2 + \phi^{-2} = \phi + 1 + 1 + 1 - \phi^{-1} = 4;$$

but 2 is the length of EB. Therefore $ABCDE$ is a regular pentagon. There is an equal pentagon for each edge of the cube, and these pentagons fit together to make the regular dodecahedron. That is, the regular dodecahedron has as vertices the 20 points

$$(\pm 1, \pm 1, \pm 1), \qquad (\pm\phi, \pm\phi^{-1}, 0), \qquad (0, \pm\phi, \pm\phi^{-1}), \qquad (\pm\phi^{-1}, 0, \pm\phi).$$

The vertices of the regular icosahedron can be understood as the 12 points

$$(\pm\phi, \pm 1, 0), \qquad (0, \pm\phi, \pm 1), \qquad (\pm 1, 0, \pm\phi);$$

indeed, the square of the distance between appropriate pairs is given by

$$\phi^2 + (\phi - 1)^2 + 1 = 2(\phi^2 - \phi + 1) = 4.$$

5. The regular polychora

We compute next

$$\Delta_4 = \sin^2\frac{\pi}{p_1} - \cos^2\frac{\pi}{p_2} - \sin^2\frac{\pi}{p_1}\cos^2\frac{\pi}{p_3} = \sin^2\frac{\pi}{p_1}\sin^2\frac{\pi}{p_3} - \cos^2\frac{\pi}{p_2}$$

which yields the condition

$$\sin\frac{\pi}{p_3}\sin\frac{\pi}{p_1} > \cos\frac{\pi}{p_2}.$$

We can understand this in terms of the *dihedral* angle of $\{p_3, p_2\}$. Let A be a vertex of this polyhedron, and let the corresponding vertex figure $\{p_2\}$ be $BCD\dots$. Let E be the foot of the perpendicular dropped from B to CA (after CA is extended if necessary). Then DE is also perpendicular to CA, and the dihedral angle of $\{p_3, p_2\}$ is BED. Let $\theta(p_3, p_2)$ be half this angle. Then

$$\sin\theta(p_3, p_2) = \frac{BD}{2BE}.$$

Let us consider the half-edge of $\{p_3, p_2\}$ as a unit. Since the polygon $BAC\dots$ is $\{p_3\}$, we have

$$BE = \sin\frac{2\pi}{p_3} = 2\sin\frac{\pi}{p_3}\cos\frac{\pi}{p_3}.$$

Since also the polygon $DAC\dots$ is $\{p_3\}$, we have

$$DC = BC = 2\cos\frac{\pi}{p_3},$$

and therefore, since the polygon BCD is $\{p_2\}$,

$$BD = 2BC\cos\frac{\pi}{p_2} = 4\cos\frac{\pi}{p_3}\cos\frac{\pi}{p_2}.$$

Putting this all together gives

$$\sin\theta(p_3, p_2) = \frac{\cos(\pi/p_2)}{\sin(\pi/p_3)}.$$

Schläfli's criterion is now

$$\sin(\pi/p_1) > \sin\theta(p_3, p_2),$$

or

$$\frac{2\pi}{p_1} > 2\theta(p_3, p_2);$$

that is, it must be possible to arrange a number p_1 of polyhedra $\{p_3, p_2\}$ so as to have a common edge.

To check when this is possible, we need:

p	$\sin\frac{\pi}{p}$	$\cos\frac{\pi}{p}$	$\frac{2\pi}{p}$
3	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	120°
4	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	90°
5	$\frac{\sqrt{(2(5-\sqrt{5}))}}{4}$	$\frac{1+\sqrt{5}}{4}$	72°

$\{p, q\}$	$\sin\theta(p, q)$	$2\theta(p, q)$
$\{3, 3\}$	$\frac{\sqrt{3}}{3}$	$\approx 70.53°$
$\{3, 4\}$	$\frac{\sqrt{6}}{3}$	$\approx 109.47°$
$\{4, 3\}$	$\frac{\sqrt{2}}{2}$	90°
$\{3, 5\}$	$\frac{\sqrt{3}+\sqrt{15}}{6}$	$\approx 138°$
$\{5, 3\}$	$\frac{\sqrt{(10(5+\sqrt{5}))}}{10}$	$\approx 116°$

Therefore the regular polychora are among

$$\{3,3,3\} \qquad \{3,3,4\}, \qquad \{4,3,3\}, \qquad \{3,3,5\}, \qquad \{5,3,3\}, \qquad \{3,4,3\};$$

but existence must be established. The first three of the possibilities are the 4-simplex, the 4-orthoplex, and the 4-cube. To establish the others, if σ is a permutation of 4, let

$$(x_0, x_1, x_2, x_3)^\sigma = (x_{\sigma(0)}, x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}).$$

5.1. **The regular 24-cell.** An instance of $\{3,4,3\}$ is the convex hull of the 24 distinct points

$$(\pm 1, \pm 1, 0, 0)^\sigma.$$

Indeed, the points of this polychoron satisfy the 24 inequalities

$$-1 \leqslant x_i \leqslant 1, \qquad \sum_{i<4} \pm x_i \leqslant 2.$$

Each hyperplane given by one of the equations $x_i = \pm 1$ or $\sum_{i<4} \pm x_i = 2$ intersects the figure in a regular octahedron of edge length $\sqrt{2}$. For example,

(1) the hyperplane defined by $x_0 = 1$ intersects the polychoron in the convex hull of the 6 points

$$(1, \pm 1, 0, 0), \qquad (1, 0, \pm 1, 0), \qquad (1, 0, 0, \pm 1);$$

(2) the hyperplane defined by $\sum_{i<4} x_i = 2$ intersects the polychoron in the convex hull of the 6 points

$$(1,1,0,0), \quad (1,0,1,0), \quad (1,0,0,1), \quad (0,1,1,0), \quad (0,1,0,1), \quad (0,0,1,1);$$

the hull in each case is a regular octahedron of edge length $\sqrt{2}$.

Every point $(\pm 1, \pm 1, 0, 0)^\sigma$ belongs to 6 of the hyperplanes so far mentioned, so it is a common vertex of 6 of the octahedrons. For example, $(1,1,0,0)$ belongs to each hyperplane defined by one of the equations

$$x_0 = 1, \qquad x_1 = 1, \qquad x_0 + x_1 \pm x_2 \pm x_3 = 2.$$

Moreover, 6 of the points $(\pm 1, \pm 1, 0, 0)^\sigma$ are at a distance of $\sqrt{2}$ from a given vertex A; each of these 6 points belongs to one of the octahedrons meeting at A; and the 6 points are the vertices of a (regular) cube. For example, $(1,1,0,0)$ is $\sqrt{2}$ away from

$$(1, 0, \pm 1, 0), \qquad (1, 0, 0, \pm 1), \qquad (0, 1, \pm 1, 0), \qquad (0, 1, 0, \pm 1).$$

So the polychoron has a vertex figure, namely a cube, at each vertex. Therefore it is a regular polychoron, namely $\{3,4,3\}$ with 24 facets.

5.2. **The regular 600-cell.**

References

[1] Jonathan Comes, *Regular polytopes*, The Montana Mathematics Enthusiast **1** (2004), no. 2, 30–37.
[2] H. S. M. Coxeter, *Regular polytopes*, third ed., Dover Publications Inc., New York, 1973. MR MR0370327 (51 #6554)
[3] John Stillwell, *The story of the 120-cell*, Notices Amer. Math. Soc. **48** (2001), no. 1, 17–24. MR MR1798928 (2001k:52019)