COUNTEREXAMPLES IN PARTIAL DERIVATIVES

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1. Introduction

If a function of two variables has both partial derivatives at a point, then the graph of the function has tangent lines in the two coordinate directions at that point, and those tangent lines span a plane. Assuming the function is f, and the point is (a, b), we can understand the plane as the graph of $L_{(a,b)}$, where

$$L_{(a,b)}(x, y) = f(a, b) + \mathrm{D}_1 f(a, b)(x - a) + \mathrm{D}_2 f(a, b)(y - b).$$

In some cases, the plane is reasonably called a **tangent plane;** in other cases, it is not. In the former cases, the function is **differentiable** at the point; in the latter, not.

Stewart's *Calculus* (5th ed., 2003), § 15.4, p. 961, gives the example of the function f given by

$$f(x, y) = \begin{cases} \dfrac{xy}{x^2 + y^2}, & \text{if } (x, y) \neq (0, 0); \\ 0, & \text{if } (x, y) = (0, 0). \end{cases}$$

Then $f(x, 0) = 0$ for all x, so $\mathrm{D}_1 f(0, 0) = 0$. By symmetry, since $f(x, y) = f(y, x)$, we have $\mathrm{D}_2 f(0, 0) = 0$. Hence $L_{(0,0)}(x, y) = 0$. But the graph of L is not tangent to the graph of f over $(0, 0)$: that is, f is not differentiable at $(0, 0)$. This is simply because f is not continuous at $(0, 0)$. This is worked out in the last two exercises of the section (41 and 42).

2. Partially differentiable, continuous, but not differentiable

It may be useful to have an example of a *continuous* function, with partial derivatives, that is not differentiable. The graph of such a function might consist of non-coplanar straight lines through the origin. Those lines would intersect the circular cylinder given by $x^2 + y^2 = 1$ at various heights. An example of such a graph can be given in **cylindrical coordinates** (§ 13.7, p. 875, of Stewart) by

$$4z = r(\sin\theta + \sin 3\theta), \tag{$*$}$$

where

$$r^2 = x^2 + y^2, \quad r\sin\theta = y, \quad r\cos\theta = x.$$

See Figure 1. The coefficient 4 is used in $(*)$ so that it will not be needed in rectangular coordinates. Indeed, we have

$$\begin{aligned} \sin\theta + \sin 3\theta &= \sin\theta + \sin 2\theta\cos\theta + \sin\theta\cos 2\theta \\ &= \sin\theta + 2\sin\theta\cos^2\theta + \sin\theta(2\cos^2\theta - 1) \\ &= 4\sin\theta\cos^2\theta, \end{aligned}$$

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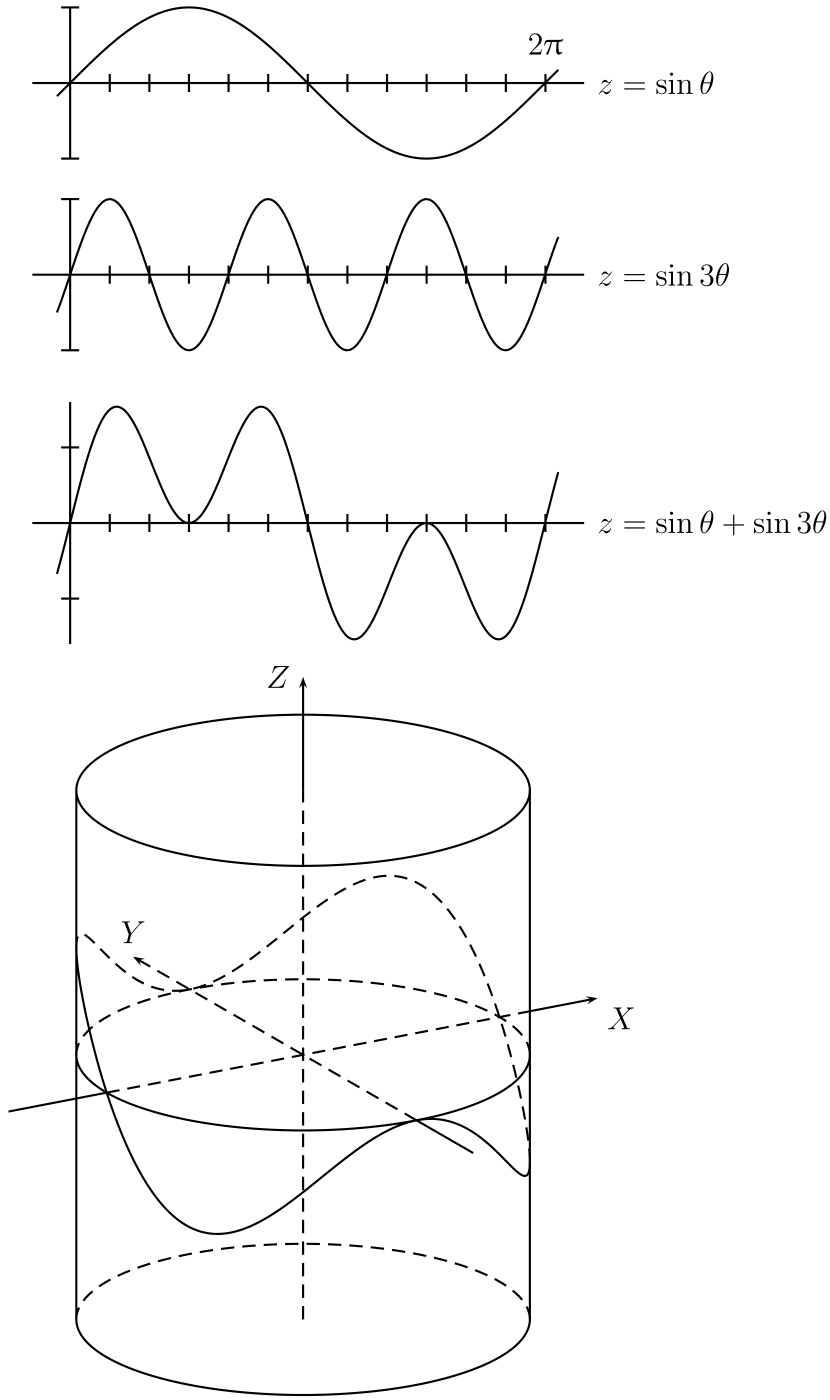

FIGURE 1

so that the graph defined by $(*)$ is the graph of the function g given by

$$g(x,y) = \begin{cases} \dfrac{x^2y}{x^2+y^2}, & \text{if } (x,y) \neq (0,0); \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

We then have

$$|g(x,y)| \leqslant |y| \leqslant \sqrt{x^2+y^2},$$

so g is continuous at $(0,0)$ by the Squeeze Theorem. Curiously, in his § 15.2, on p. 941, Stewart mentions that the Squeeze Theorem holds for functions of several variables, though without making a formal statement of the theorem in this context; then he immediately proves in Example 4 that $\lim_{(x,y)\to(0,0)} 3x^2y/(x^2+y^2) = 0$, but *without* using the Squeeze Theorem! The possibility of using the Squeeze Theorem in his example is relegated to a marginal note.

In our present example, we still have $\mathrm{D}_1\, g(0,0) = 0 = \mathrm{D}_2\, g(0,0)$, as in the example of § 1. However, since $g(x,x) = x/2$, the graph of g contains a line through the origin that is not horizontal. Therefore g is not differentiable at $(0,0)$. Stewart says, as Theorem 15.4.8, that *continuity* of the partial derivatives is sufficient to ensure differentiability. The proof is in an appendix. We may confirm that this condition fails in the present example: When $(x,y) \neq (0,0)$, then

$$\mathrm{D}_1\, g(x,y) = \frac{2xy(x^2+y^2) - 2x(x^2y)}{(x^2+y^2)^2} = \frac{2xy^3}{(x^2+y^2)^2}, \qquad \mathrm{D}_1\, g(x,x) = \frac{1}{2};$$
$$\mathrm{D}_2\, g(x,y) = \frac{x^2(x^2+y^2) - 2y(x^2y)}{(x^2+y^2)^2} = \frac{x^2(x^2-y^2)}{(x^2+y^2)^2}, \qquad \mathrm{D}_2\, g(x,0) = 1.$$

3. Different mixed partials

In his § 15.3, p. 952, Stewart states **Clairaut's Theorem** on the equality of mixed partial derivatives. Again, the proof is in an appendix. Nowhere (as far as I can tell) is an example offered where the hypotheses and conclusion of the theorem fail. Yet I found students to be curious about such an example. Adams's *Calculus: a complete course* (4th ed., 1999) gives an example in an exercise (12.4.16, p. 720), without suggesting how the example might be derived.

Yet the example can be derived as g was above. We look for a function h meeting two conditions:

(1) The graph of h has the tangent plane (above the origin) given by $z = 0$.
(2) The intersection of the graph of h with a circular cylinder given by $x^2 + y^2 = a^2$ is given by $z = b\sin 4\theta$ for some b depending on a. This should ensure that $\mathrm{D}_2\, f(x,0)$ increases as x does, but $\mathrm{D}_1\, f(0,y)$ decreases as y increases: see Figure 2.

Our two conditions are met when h has the graph given by

$$2z = r^2 \sin 4\theta.$$

Since $\sin 4\theta = 2\sin 2\theta \cos 2\theta = 2\sin\theta\cos\theta(\cos^2\theta - \sin^2\theta)$, we have

$$h(x,y) = \begin{cases} \dfrac{xy(x+y)(x-y)}{x^2+y^2} = \dfrac{xy(x^2-y^2)}{x^2+y^2} = \dfrac{x^3y - xy^3}{x^2+y^2}, & \text{if } (x,y) \neq (0,0); \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

Away from $(0,0)$, we have

$$\begin{aligned} \mathrm{D}_1\, h(x,y) &= y \cdot \frac{(3x^2-y^2)(x^2+y^2) - 2x(x^3-xy^2)}{(x^2+y^2)^2} \\ &= y \cdot \frac{x^4 + 4x^2y^2 - y^4}{(x^2+y^2)^2}, \end{aligned}$$

and therefore

$$\mathrm{D}_1\, h(0,y) = -y.$$

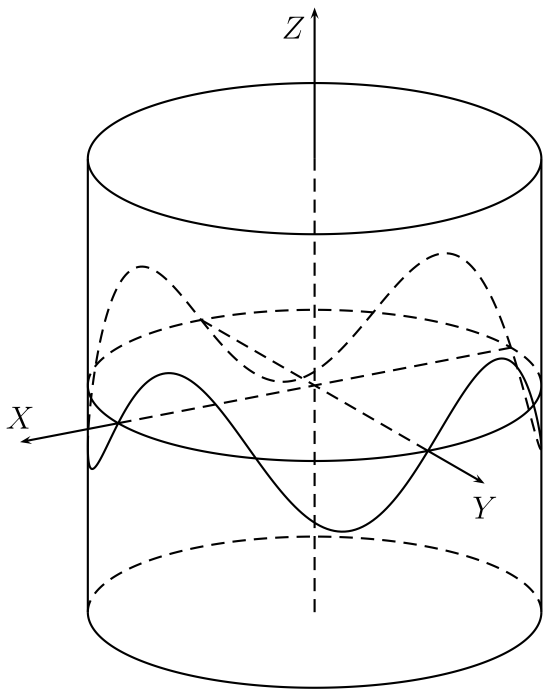

FIGURE 2

This holds even when $y = 0$, and so

$$D_2 D_1 h(0, y) = -1.$$

By symmetry, since $h(y, x) = -h(x, y)$, we have

$$D_2 h(x, y) = -D_1 h(y, x) = x \cdot \frac{x^4 + 4x^2y^2 - y^4}{(x^2 + y^2)^2},$$

$$D_2 h(x, 0) = x,$$

$$D_1 D_2 h(x, 0) = 1,$$

$$D_1 D_2 h(0, 0) = 1 \neq D_2 D_1 h(0, 0).$$

To confirm that the mixed partials are not continuous, we can compute, when $(x, y) \neq (0, 0)$,

$$D_1 h(x, y) = \frac{x^4y + 4x^2y^3 - y^5}{(x^2 + y^2)^2},$$

$$D_2 D_1 h(x, y) = \frac{(x^4 + 12x^2y^2 - 5y^4)(x^2 + y^2) - 4y(x^4y + 4x^2y^3 - y^5)}{(x^2 + y^2)^3}$$

$$= \frac{x^6 + 9x^4y^2 - 9x^2y^4 - y^6}{(x^2 + y^2)^3},$$

$$D_2 D_1 h(x, 0) = 1 \neq -1 = D_2 D_1 h(0, 0).$$

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