A DERIVATION OF THE EQUATION $\mathrm{e}^{\pi\mathrm{i}} + 1 = 0$

DAVID PIERCE

Contents

0. Introduction

The equation of the title, which is (2) below, is known as (a special case of) Euler's Formula. It combines the five constants 0, 1, π, e, and i (that is, $\sqrt{-1}$) by means of the three operations of addition, multiplication, and exponentiation, and the relation of equality. The equation arises from an analysis of the complex numbers. I aim here to derive the equation for somebody with some experience in calculus. However, the argument is condensed, and will require work on the part of the reader.

In § 2, I shall use some notation that is not so common as it might be [2, ch. 3, p. 45]. If one has an expression involving x, perhaps $x^3 - 14 \cdot \sin\log x$, then one sometimes speaks of the 'function'

$$f(x) = x^3 - 14 \cdot \sin\log x, \tag{1}$$

or even the 'function' $y = x^3 - 14 \cdot \sin\log x$. However, these are not strictly functions; they are equations. The latter equation does *define* a function, namely

$$x \longmapsto x^3 - 14 \cdot \sin\log x.$$

This is the same as the function $y \mapsto y^3 - 14 \cdot \sin\log y$, the function $z \mapsto z^3 - 14 \cdot \sin\log z$, and so forth; repetition *binds* the variable in each case. We may give this function a one-letter name like f; in that case, (1) can be understood as a defining identity for f.

The derivative—if it exists—of a function f is the function denoted by f', namely

$$x \longmapsto \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

One may also write $\mathrm{d}\,f(x)/\,\mathrm{d}\,x$ for the derivative of f. But I shall avoid this notation, because it is used sloppily, as (1) is [2, ch. 9, p. 140]. One tends to write, for example, $\mathrm{d}\,x^2/\,\mathrm{d}\,x = 2x$; that is, one treats $\mathrm{d}\,f(x)/\,\mathrm{d}\,x$ as $f'(x)$, the derivative of f evaluated at x. Then one must distinguish $\mathrm{d}\,f(x)/\,\mathrm{d}\,x$ from $\mathrm{d}\,f(y)/\,\mathrm{d}\,y$, and one does not have an obvious related notation for $f'(a)$ (although $(\mathrm{d}\,f(x)/\,\mathrm{d}\,x)\big|_{x=a}$ is used). It would be more precise

Date: March 17, 2006; revised, December 24, 2007.

to write, for example, $\mathrm{d}\,x^2/\,\mathrm{d}\,x = (x \mapsto 2x)$, or even $\mathrm{d}\,x^2/\,\mathrm{d}\,x = (y \mapsto 2y)$. Then $f'(a)$ could be written as $(\mathrm{d}\,f(x)/\,\mathrm{d}\,x)(a)$.

1. Bare Bones

Use power-series expansions for the exponential, sine, and cosine functions:

$$\mathrm{e}^x = \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

and therefore

$$\begin{aligned}
\mathrm{e}^{x\mathrm{i}} &= \sum_{n=0}^{\infty} \frac{(x\mathrm{i})^n}{n!} \\
&= \sum_{m=0}^{\infty} \frac{(x\mathrm{i})^{2m}}{(2m)!} + \sum_{m=0}^{\infty} \frac{(x\mathrm{i})^{2m+1}}{(2m+1)!} \\
&= \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{(2m)!} + \mathrm{i} \cdot \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{(2m+1)!} \\
&= \cos x + \mathrm{i} \cdot \sin x.
\end{aligned}$$

Thus the general Euler's formula:

$$\mathrm{e}^{\mathrm{i}\cdot x} = \cos x + \mathrm{i} \cdot \sin x.$$

Now let x be π; the result is $\mathrm{e}^{\pi\mathrm{i}} = -1$, or

$$\mathrm{e}^{\pi\mathrm{i}} + 1 = 0. \tag{2}$$

2. Sinews

2.1. The real and the complex numbers. The real numbers compose a set $\mathbb{R}$; the complex, $\mathbb{C}$. One writes a typical complex number as

$$x + \mathrm{i} \cdot y$$

(or $x + y\mathrm{i}$), where x and y stand for real numbers. One does arithmetic with complex numbers according to the usual rules; but one can always replace i^2 with -1. One can identify $x + \mathrm{i} \cdot y$ with the point (x, y) of the Cartesian plane, $\mathbb{R} \times \mathbb{R}$ (Fig. 1); in particular, a real number x is identified with the point $(x, 0)$ on the X-axis, and i is identified with $(0, 1)$ on the Y-axis. The rule of multiplication in $\mathbb{R} \times \mathbb{R}$ is then

$$(x, y) \cdot (u, v) = (xu - yv,\ xv + yu).$$

This can be obtained from a multiplication of matrices:

$$\begin{pmatrix} x & y \\ -y & x \end{pmatrix} \cdot \begin{pmatrix} u & v \\ -v & u \end{pmatrix} = \begin{pmatrix} xu - yv & xv + yu \\ -yu - xv & -yv + xu \end{pmatrix}.$$

Also, let us assign the following names:

$$(0, 0) = O, \quad (1, 0) = A, \quad (x, y) = P, \quad (u, v) = Q, \quad (xu - yv, xv + yu) = R.$$

It will be shown in the next sub-section that

$$\angle AOP + \angle AOQ = \angle AOR, \qquad |OP| \cdot |OQ| = |OR|.$$

Thus complex numbers provide a notation for certain geometrical effects.

2.2. Trigonometric functions. The geometric meaning of complex multiplication can be seen with trigonometry.

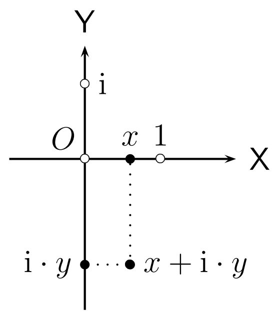

FIGURE 1. The Cartesian plane as $\mathbb{C}$.

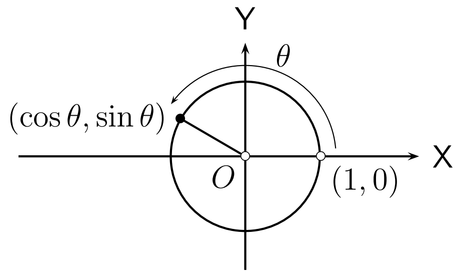

FIGURE 2. Definition of sine and cosine.

2.2.1. *Definitions.* The trigonometric functions of sine and cosine can be defined in terms of the unit circle, which is given by

$$x^2 + y^2 = 1. \tag{3}$$

An arbitrary point on this circle has coordinates

$$(\cos\theta,\ \sin\theta),$$

where θ is the distance along the circle, in the counterclockwise direction, from $(1, 0)$ to the point (Fig. 2). Alternatively, θ here is half the area of the sector of the circle bounded by the radii to $(1, 0)$ and the other point. Then θ is also considered as the size, in 'radians', of the angle filled by this sector.

From (3) now follows the Pythagorean identity

$$\sin^2 + \cos^2 = 1. \tag{4}$$

Also, by definition,

$$\pi$$

is the distance halfway around the circle; so we have a table of values:

θ	0	$\pi/2$	π	$3\pi/2$	2π
$\sin\theta$	0	1	0	-1	0
$\cos\theta$	1	0	-1	0	1

2.2.2. *Complex multiplication.* Again, we identify $(\cos\theta, \sin\theta)$ with the complex number

$$\cos\theta + \mathrm{i} \cdot \sin\theta.$$

The claim of § 2.1 is that to multiply by this number is to rotate (counterclockwise, about O or 0) through the angle of size θ. To prove this, it is enough to observe that:

(1) the claim is true when the multiplicand is 1 or i;
(2) $z \cdot (x + \mathrm{i} \cdot y) = (z \cdot 1)x + (z \cdot \mathrm{i})y$.

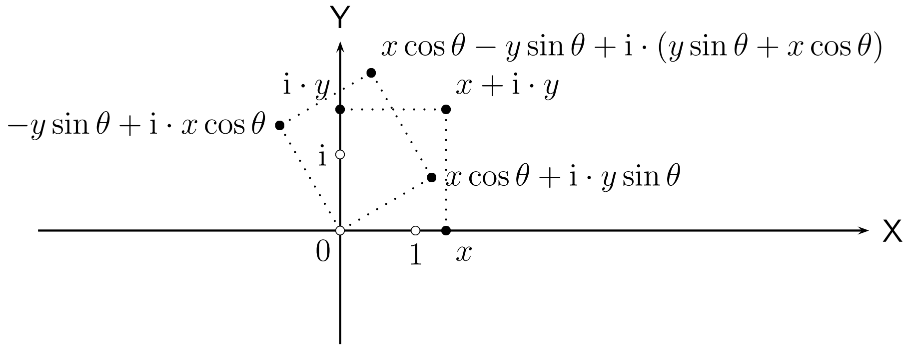

FIGURE 3. Complex multiplication.

(See Fig. 3.)

2.2.3. *Angle addition formulas.* To rotate by α, then β, is to rotate by $\alpha+\beta$. Hence

$$\begin{aligned}\cos(\alpha+\beta)+\mathrm{i}\cdot\sin(\alpha+\beta) &= (\cos\alpha+\mathrm{i}\cdot\sin\alpha)\cdot(\cos\beta+\mathrm{i}\cdot\sin\beta)\\ &= \cos\alpha\cos\beta-\sin\alpha\sin\beta+\mathrm{i}\cdot(\sin\alpha\cos\beta+\cos\alpha\sin\beta),\end{aligned}$$

yielding the identities

$$\cos(\alpha+\beta)=\cos\alpha\cdot\cos\beta-\sin\alpha\cdot\sin\beta, \tag{5}$$
$$\sin(\alpha+\beta)=\sin\alpha\cdot\cos\beta+\cos\alpha\cdot\sin\beta. \tag{6}$$

2.2.4. *Derivatives at* 0. One can observe geometrically that

$$\lim_{\theta\to0}\frac{\sin\theta}{\theta}=1 \tag{7}$$

(see Fig. 4.) Indeed, if $0<\theta<\pi/2$, then $\sin\theta$ is the length of the straight line between $(\cos\theta,0)$ and $(\cos\theta,\sin\theta)$; in this case, this length is bounded by two arcs of circles, one of radius $\cos\theta$, the other, 1; in particular,

$$\theta\cos\theta<\sin\theta<\theta.$$

(One can also understand this as relating the *areas* of two sectors and a triangle.) Therefore

$$\cos\theta<\frac{\sin\theta}{\theta}<1.$$

This holds also if $-\pi/2<\theta<0$. Since $\lim_{\theta\to0}\cos\theta=1$, we have (7) by the 'Squeeze Theorem'.

By the definition of derivative, (7) means

$$\sin' 0=1. \tag{8}$$

Similarly, and by (4), we have

$$\begin{aligned}\lim_{\theta\to0}\frac{\cos\theta-1}{\theta}=\lim_{\theta\to0}\frac{\cos^2\theta-1}{\theta(\cos\theta+1)}&=\lim_{\theta\to0}\frac{-\sin^2\theta}{\theta(\cos\theta+1)}\\ &=-\lim_{\theta\to0}\frac{\sin\theta}{\theta}\cdot\lim_{\theta\to0}\frac{\sin\theta}{\cos\theta+1}=-1\cdot0=0,\end{aligned}$$

which means

$$\cos' 0=0. \tag{9}$$

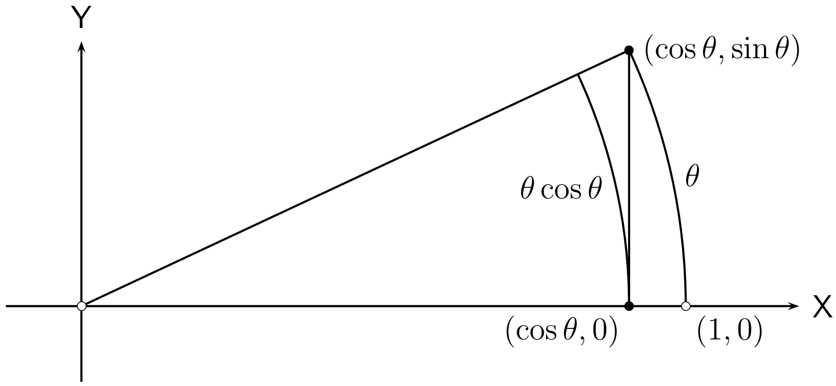

FIGURE 4. $\sin\theta \approx \theta$ when $|\theta|$ is small.

2.2.5. *Derivatives.* Using (5) and (6), take the derivatives of $x \mapsto \cos(\alpha + x)$ and $x \mapsto \sin(\alpha + x)$ and evaluate at 0 by means of (8) and (9):

$$\cos'\alpha = \cos'(\alpha + 0) = \cos\alpha \cdot \cos' 0 - \sin\alpha \cdot \sin' 0 = -\sin\alpha, \tag{10}$$

$$\sin'\alpha = \sin'(\alpha + 0) = \sin\alpha \cdot \cos' 0 + \cos\alpha \cdot \sin' 0 = \cos\alpha. \tag{11}$$

Thus both sin and cos are solutions of the differential equation

$$y'' + y = 0.$$

2.3. **The logarithm function.** The function $x \mapsto \log x$ is the unique function f defined on the interval $(0, \infty)$—namely, $\{x \in \mathbb{R} \colon 0 < x < \infty\}$—such that:

(1) $f'(x) = 1/x$ whenever $0 < x < \infty$;
(2) $f(1) = 0$.

This is the logarithm function. It sometimes denoted by $x \mapsto \ln x$, for *logarithmus naturalis,* but the mathematician works only with natural things anyway, so the 'n' is redundant. (According to [1] *Oxford English Dictionary* [2nd ed., 1989], John Napier of Merchiston, in 1614, coined the Latin word *logarithmus,* from the Greek words λόγος and ἀριθμός, as a name for the function we have just defined.)

Since it has a positive derivative, log is an increasing function; in particular, it is invertible.

The range of log is all of $\mathbb{R}$ (see Fig. 6). Indeed, by the Mean Value Theorem, since $\log' x = 1/x$, if j is a positive integer, then

$$\log 2^{j+1} - \log 2^j = 2^j \cdot \frac{\log 2^{j+1} - \log 2^j}{2^{j+1} - 2^j} = 2^j \cdot \frac{1}{c} \geqslant \frac{1}{2}$$

for some c in the interval $(2^j, 2^{j+1})$. Hence

$$\log 2^n = \sum_{j=0}^{n-1} (\log 2^{j+1} - \log 2^j) \geqslant \frac{n}{2},$$

so $\lim_{x\to\infty} \log x = \infty$. Similarly,

$$\log 2^{-n} = \sum_{j=0}^{n-1} (\log 2^{-(j+1)} - \log 2^{-j}) \leqslant \frac{-n}{2},$$

so $\lim_{x\to 0^+} \log x = -\infty$.

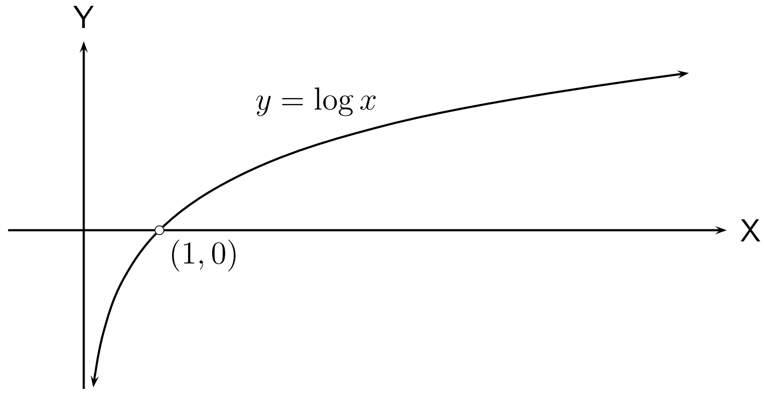

FIGURE 5. The logarithm function.

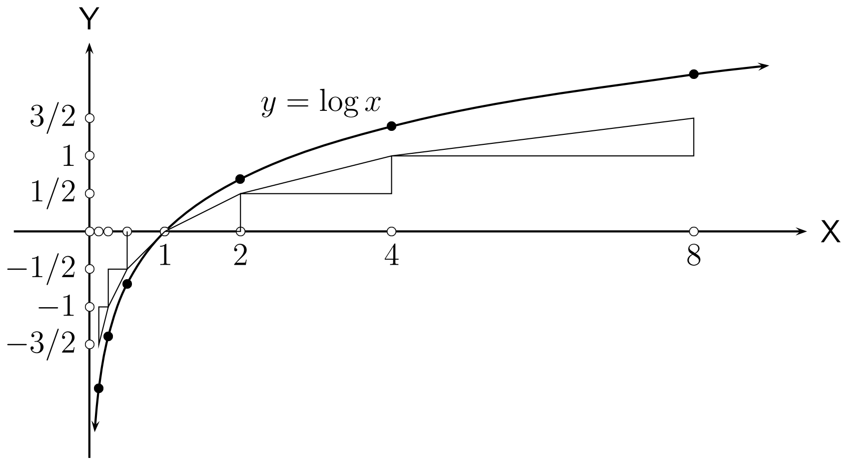

FIGURE 6. The range of log is $\mathbb{R}$.

2.4. **The exponential function.** The inverse of log is denoted by exp, so that

$$\exp 0 = 1. \tag{12}$$

Since the range of log is $\mathbb{R}$, so is the domain of exp. Therefore

$$x = \log \exp x \tag{13}$$

for all x in $\mathbb{R}$. Also,

$$x = \exp \log x \tag{14}$$

for all positive x. From (13), we can compute the derivative of exp by means of the Chain Rule:

$$1 = (\log \circ \exp)' x = \log'(\exp x) \cdot \exp' x = \frac{1}{\exp x} \cdot \exp' x,$$

$$\exp' x = \exp x. \tag{15}$$

Thus exp is a solution of the differential equation

$$y' = y.$$

The notation e^x for $\exp x$ is justified as follows. When n is a positive integer, then

$$x^n = \underbrace{x \cdots x}_{n}.$$

Also, $x^0 = 1$, and $x^{-n} = 1/x^n$, and $x^{1/n} = \sqrt[n]{x}$, and $x^{m/n} = (x^{1/n})^m$ if m is another integer. So x^a is defined for all *rational* numbers a (at least if $x > 0$). Also, the derivative of $x \mapsto x^a$ is $x \mapsto ax^{a-1}$. Now let f_a be the function $x \mapsto \log(x^a)$. Then

$$f_a{}'(x) = \frac{1}{x^a} \cdot ax^{a-1} = \frac{a}{x} = (a \log)'x,$$

but also $f_a(1) = 0 = a \log 1$; therefore $f_a = a \log$. Applying exp and using (14) gives

$$x^a = \exp(a \log x). \tag{16}$$

We can take this as a *definition* of x^a when a is irrational. We also define

$$\mathrm{e} = \exp 1,$$

so that $\log \mathrm{e} = 1$. Then (16) yields

$$\mathrm{e}^x = \exp(x \log \mathrm{e}) = \exp x. \tag{17}$$

2.5. **Power-series expansions.** If f is a polynomial function $x \mapsto \sum_{i=0}^{n} a_i x^i$, then

$$f(0) = a_0, \quad f'(0) = a_1, \quad f''(0) = 2a_2, \quad f'''(0) = 6a_3, \quad \dots,$$

$$f^{(m)}(0) = \begin{cases} m!\, a_m, & \text{if } m \leqslant n; \\ 0, & \text{if } m > n. \end{cases}$$

Hence

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k.$$

One expects reasonable functions to be approximable by polynomials, in particular, by polynomials that share their first several derivatives at 0 (or some other point). More precisely, if f is an arbitrary differentiable function, then one expects

$$f(x) \approx \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k,$$

with the approximation improving as n grows larger, so that, 'in the limit,'

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n. \tag{18}$$

The expectation is not always realized; it *is* realized when $f = \exp$; by (12) and (15), we have

$$\exp x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots. \tag{19}$$

We can use (19) to *define* $\exp x$ when x is a complex number.

As more special cases of (18), by (8), (9), (10), and (11), we have

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}, \tag{20}$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!} x^{2n+1}. \tag{21}$$

From (19),(20), and (21), we have

$$\exp(\mathrm{i} \cdot x) = \cos x + \mathrm{i} \cdot \sin x.$$

In particular, $\exp(\mathrm{i}\pi) = -1$, which by (17) yields (2).

References

[1] John Simpson et al., editors. *Oxford English Dictionary.* Oxford University Press, 2nd edition, 1989. `http://dictionary.oed.com/` (accessed December 24, 2007).

[2] Michael Spivak. *Calculus. 2nd ed.* Berkeley, California: Publish Perish, Inc. XIII, 647 pp., 1980.

Mathematics Dept, Middle East Tech. Univ., Ankara 06531, Turkey
E-mail address: `dpierce@metu.edu.tr`
URL: `http://www.math.metu.edu.tr/~dpierce/`