The Binomial Theorem

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I prepared these notes after teaching from § 12.11 'The Binomial Series' and § 12.12 'Applications of Taylor Polynomials' of Stewart's Calculus*, 5th ed., 2003.*

The Binomial Theorem learned in high-school is

$$(1+x)^k = \sum_{n=0}^{k} \binom{k}{n} x^n,$$

where k is a positive integer or 0, and

$$\binom{k}{n} = \begin{cases} 1, & \text{if } n = 0; \\ \dfrac{k}{n} \cdot \dfrac{k-1}{n-1} \cdots \dfrac{k-n+1}{1}, & \text{if } n > 0. \end{cases}$$

This definition of $\binom{k}{n}$ makes sense for *all* real numbers k. In particular, if k is an integer, and $0 \leqslant k < n$, then $\binom{k}{n} = 0$. Therefore, again if k is a positive integer or 0, we have

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n. \tag{$*$}$$

Since the series here has the form of a Maclaurin series, it must *be* the Maclaurin series for $(1+x)^k$, and its radius of convergence is ∞. In particular, if $f(x) = (1+x)^k$, then

$$\frac{f^{(n)}(0)}{n!} = \binom{k}{n}.$$

This equation also holds for all real k by direct computation:

$$\begin{aligned} f'(x) &= k(1+x)^{k-1}, \\ f''(x) &= k(k-1)(1+x)^{k-2}, \\ &\cdots \\ f^{(n)}(x) &= k(k-1)\cdots(k-n+1)(1+x)^{k-n}. \end{aligned}$$

So the series in $(*)$ is still the Maclaurin series for $(1+x)^k$. We ask:

(i) what is the radius R of convergence of this series?

(ii) when $|x| < R$, does $(*)$ hold?

To answer (i), use the Ratio Test: if $a_n = \binom{k}{n} x^n$, then

$$\frac{a_{n+1}}{a_n} = \frac{k(k-1)\cdots(k-n)}{(n+1)!} \cdot \frac{n!}{k(k-1)\cdots(k-n+1)} \cdot \frac{x^{n+1}}{x^n}$$
$$= \frac{k-n}{n+1} \cdot x.$$

Since

$$\lim_{n\to\infty} \left| \frac{k-n}{n+1} \cdot x \right| = |x| \cdot \lim_{n\to\infty} \left| \frac{-n+k}{n+1} \right| = |x|,$$

we have $R = 1$. So when $|x| < 1$, then

$$\sum_{n=0}^{\infty} \binom{k}{n} x^n = g_k(x)$$

for some function $g_k(x)$. But have we $g_k(x) = (1+x)^k$? This is question (ii). To answer it, note first that, if $n > 0$, then

$$\binom{k-1}{n} = \frac{k-1}{n}\frac{k-2}{n-1}\cdots\frac{k-n}{1}$$
$$= \frac{k-n}{n}\frac{k-1}{n-1}\cdots\frac{k-n+1}{1} = \frac{k-n}{n}\binom{k-1}{n-1},$$

so that

$$\binom{k-1}{n} + \binom{k-1}{n-1} = \left(\frac{k-n}{n} + 1\right)\binom{k-1}{n-1}$$
$$= \frac{k}{n}\binom{k-1}{n-1} = \binom{k}{n}.$$

Therefore

$$
\begin{aligned}
(1+x)g_{k-1}(x) &= (1+x)\sum_{n=0}^{\infty}\binom{k-1}{n}x^n \\
&= \sum_{n=0}^{\infty}\binom{k-1}{n}x^n + \sum_{n=0}^{\infty}\binom{k-1}{n}x^{n+1} \\
&= 1 + \sum_{n=1}^{\infty}\binom{k-1}{n}x^n + \sum_{n=1}^{\infty}\binom{k-1}{n-1}x^n \\
&= 1 + \sum_{n=1}^{\infty}\binom{k}{n}x^n = \sum_{n=0}^{\infty}\binom{k}{n}x^n = g_k(x).
\end{aligned}
$$

Now we can compute

$$
\begin{aligned}
g_k{}'(x) &= \sum_{n=0}^{\infty}\binom{k}{n}nx^{n-1} = \sum_{n=1}^{\infty}\binom{k}{n}nx^{n-1} \\
&= \sum_{n=0}^{\infty}\binom{k}{n+1}(n+1)x^n = k\sum_{n=0}^{\infty}\binom{k}{n+1}\frac{n+1}{k}\cdot x^n \\
&= k\sum_{n=0}^{\infty}\binom{k-1}{n}x^n = k\cdot g_{k-1}(x).
\end{aligned}
$$

If we define

$$
h(x) = \frac{g_k(x)}{(1+x)^k},
$$

then $h(0) = 1$, and

$$
h'(x) = \frac{g_k{}'(x)(1+x)^k - k(1+x)^{k-1}g_k(x)}{(1+x)^{2k}} = 0.
$$

Hence h is constant, and $h(x) = 1$ whenever $|x| < 1$, and then $g_k(x) = (1+x)^k$. Thus, $(*)$ holds for all x, when k is a positive integer or 0, and when $|x| < 1$, otherwise. For example,

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} \binom{-1}{n} (-x)^n = \sum_{n=0}^{\infty} \frac{-1}{n} \frac{-2}{n-1} \cdots \frac{-n}{1} (-1)^n x^n$$
$$= \sum_{n=0}^{\infty} x^n,$$

as we expect. Also

$$\sqrt{1+x} = \sum_{n=0}^{\infty} \binom{1/2}{n} x^n.$$

Since

$$\binom{k}{n+1} = \binom{k}{n} \frac{k-n}{n+1},$$

we have

$$\binom{1/2}{0} = 1, \qquad \binom{1/2}{1} = \frac{1}{2},$$

and then

$$\binom{1/2}{2} = \frac{1}{2} \cdot \frac{-1/2}{2} = -\frac{1}{8},$$
$$\binom{1/2}{3} = -\frac{1}{8} \cdot \frac{-3/2}{3} = \frac{1}{16},$$
$$\binom{1/2}{4} = \frac{1}{16} \cdot \frac{-5/2}{4} = -\frac{5}{128},$$

and generally

$$\binom{1/2}{n} = (-1)^{n+1} \cdot \frac{1 \cdot 3 \cdots (2n-3)}{2^n n!}$$

when $n > 1$. Therefore

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \cdots,$$

an alternating series when $x > 0$. In particular,

$$\sqrt{1+x} \approx 1 + \frac{x}{2}$$

when $|x|$ is small (this is the tangent-line approximation); to be precise, if $0 < x < 1$, then

$$0 < \sqrt{1+x} - \left(1 + \frac{x}{2}\right) \leqslant \frac{x^2}{8}.$$

For a better approximation, again when $0 < x < 1$,

$$\left|\sqrt{1+x} - \left(1 + \frac{x}{2} - \frac{1}{8}x^2\right)\right| \leqslant \frac{x^3}{16}.$$

In Einstein's special theory of relativity,

$$m = \frac{m_0}{\sqrt{1 - v^2/\mathrm{c}^2}},$$

where m is the mass of an object with velocity v whose rest mass is m_0, and c is the speed of light. The kinetic energy K

of the object is then given by

$$K = (m - m_0)\mathrm{c}^2 = m_0\mathrm{c}^2\left(\frac{1}{\sqrt{1 - v^2/\mathrm{c}^2}} - 1\right)$$
$$= m_0\mathrm{c}^2\left(\left(1 - \frac{v^2}{\mathrm{c}^2}\right)^{-1/2} - 1\right)$$
$$= m_0\mathrm{c}^2\left(\sum_{n=0}^{\infty}\binom{-1/2}{n}\left(\frac{-v^2}{\mathrm{c}^2}\right)^n - 1\right)$$
$$= m_0\mathrm{c}^2\sum_{n=1}^{\infty}(-1)^n\binom{-1/2}{n}\left(\frac{v^2}{\mathrm{c}^2}\right)^n$$
$$= m_0\mathrm{c}^2\left(\frac{v^2}{2\mathrm{c}^2} + \frac{3v^2}{8\mathrm{c}^2} + \cdots\right).$$

In particular, for low velocities,

$$K \approx \frac{1}{2}m_0v^2.$$

To be precise, by Taylor's Theorem,

$$f(x) = f(0) + f'(0)\cdot x + \frac{1}{2}f''(t)\cdot x^2$$

for some t between x and 0. If $f(x) = (1 + x)^{-1/2}$, then $f''(x) = 3(1 + x)^{-5/2}/4$, so that, if $-1 < x < 0$, then also $x < t < 0$, and

$$|f(x) - f(0) - f'(0)x| = \frac{1}{2}\left|f''(t)x^2\right| < \frac{3x^2}{8(1 + x)^{5/2}}.$$

In our case, letting $x = -v^2/\mathrm{c}^2$, we have

$$0 < K - \frac{1}{2}m_0v^2 < m_0\mathrm{c}^2\cdot\frac{3(v^2/\mathrm{c}^2)^2}{8(1 - v^2/\mathrm{c}^2)^{5/2}}.$$