Number theory exercises

MAT 221, fall 2014

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Exercise 1. Assuming $b > 1$, show $\dfrac{\sigma(a)}{a} < \dfrac{\sigma(ab)}{ab} \leqslant \dfrac{\sigma(a) \cdot \sigma(b)}{ab}$.

Exercise 2. Show that

a) an odd number with no more than two prime factors cannot be perfect;

b) if a is perfect, then $\displaystyle\sum_{d \mid a} \frac{1}{d} = 2$.

Exercise 3. Prove the following.

a) $\phi(2n) = \phi(n)$ if n is odd.

b) $\phi(2^m) = \begin{cases} 2^{m-1}, & \text{if } m > 0, \\ 1, & \text{if } m = 0. \end{cases}$

c) If $k > 0$ and $\ell > 0$, then $\phi(2^k \cdot 3^\ell) \mid 2^k \cdot 3^\ell$.

d) If $\phi(a) \mid a$, then a is of the form 1 or 2^k or $2^k \cdot 3^\ell$, where $k > 0$ and $\ell > 0$.

Exercise 4. Prove the converse of Möbius Inversion Theorem; that is, for arithmetic functions F and G, prove

$$F(a) = \sum_{d|a} \mu(d) \cdot G\left(\frac{a}{d}\right) \implies G(a) = \sum_{d|a} F(d).$$

Exercise 5. a) Show $\sum_{d|a} \mu(d) \cdot \sigma\left(\frac{a}{d}\right) = a.$

b) Defining $\Lambda(a) = \log p$ if $a = p^c$, where $c > 0$, and otherwise $\Lambda(a) = 0$, show

$$\log a = \sum_{d|a} \Lambda(d), \qquad \Lambda(a) = -\sum_{d|a} \mu(d) \cdot \log d.$$

Exercise 6. If $0 < k < p$, show that $p \mid \binom{p}{k}$.

Exercise 7. a) If $a \equiv b \pmod{m}$ and $a \equiv b \pmod{n}$, show

$$a \equiv b \pmod{\operatorname{lcm}(m, n)}.$$

b) If $k > 0$ and a is odd, show $a^{2^k} \equiv 1 \pmod{2^{k+2}}$.

c) If p is an odd prime, let $\kappa(p^\ell) = \phi(p^\ell)$. Also let

$$\kappa(2^\ell) = \begin{cases} \phi(2^\ell), & \text{if } 0 \leqslant \ell \leqslant 2, \\ \phi(2^\ell)/2, & \text{if } \ell > 2. \end{cases}$$

Now define generally

$$\kappa(a) = \operatorname*{lcm}_{p|a} \kappa(p^{a(p)}).$$

If $\gcd(a, m) = 1$, show $a^{\kappa(m)} \equiv 1 \pmod{m}$.

Exercise 8. Solve Chinese Remainder Theorem problems.