THE THIRTEEN BOOKS OF EUCLID'S ELEMENTS

TRANSLATED FROM THE TEXT OF HEIBERG

WITH INTRODUCTION AND COMMENTARY

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BOOK VII.

DEFINITIONS.

1. An **unit** is that by virtue of which each of the things that exist is called one.

2. A **number** is a multitude composed of units.

3. A number is **a part** of a number, the less of the greater, when it measures the greater;

4. but **parts** when it does not measure it.

5. The greater number is a **multiple** of the less when it is measured by the less.

6. An **even number** is that which is divisible into two equal parts.

7. An **odd number** is that which is not divisible into two equal parts, or that which differs by an unit from an even number.

8. An **even-times even number** is that which is measured by an even number according to an even number.

9. An **even-times odd number** is that which is measured by an even number according to an odd number.

10. An **odd-times odd number** is that which is measured by an odd number according to an odd number.

11. A **prime number** is that which is measured by an unit alone.

12. Numbers **prime to one another** are those which are measured by an unit alone as a common measure.

13. A **composite number** is that which is measured by some number.

14. Numbers **composite to one another** are those which are measured by some number as a common measure.

15. A number is said to **multiply** a number when that which is multiplied is added to itself as many times as there are units in the other, and thus some number is produced.

16. And, when two numbers having multiplied one another make some number, the number so produced is called **plane**, and its **sides** are the numbers which have multiplied one another.

17. And, when three numbers having multiplied one another make some number, the number so produced is **solid**, and its **sides** are the numbers which have multiplied one another.

18. A **square number** is equal multiplied by equal, or a number which is contained by two equal numbers.

19. And a **cube** is equal multiplied by equal and again by equal, or a number which is contained by three equal numbers.

20. Numbers are **proportional** when the first is the same multiple, or the same part, or the same parts, of the second that the third is of the fourth.

21. **Similar plane** and **solid** numbers are those which have their sides proportional.

22. A **perfect number** is that which is equal to its own parts.

DEFINITION 1.

Μονάς ἐστιν, καθ' ἣν ἕκαστον τῶν ὄντων ἓν λέγεται.

Iamblichus (fl. *circa* 300 A.D.) tells us (*Comm. on Nicomachus*, ed. Pistelli, p. 11, 5) that the Euclidean definition of an *unit* or a *monad* was the definition given by "more recent" writers (οἱ νεώτεροι), and that it lacked the words "even though it be collective" (κἂν συστηματικὸν ᾖ). He also gives (*ibid.* p. 11) a number of other definitions. (1) According to "some of the Pythagoreans," "an unit is the boundary between number and parts" (μονάς ἐστιν ἀριθμοῦ καὶ μορίων μεθόριον), "because from it, as from a seed and eternal root, ratios increase reciprocally on either side," i.e. on one side we have multiple ratios continually increasing and on the other (if the unit be subdivided) submultiple ratios with denominators continually increasing. (2) A somewhat similar definition is that of Thymaridas, an ancient Pythagorean, who defined a monad as "limiting quantity" (περαίνουσα ποσότης), the beginning and the end of a thing being equally an extremity (πέρας). Perhaps the words together with their explanation may best be expressed by "limit of fewness." Theon of Smyrna (p. 18, 6, ed. Hiller) adds the explanation that the monad is "that which, when the multitude is diminished by way of continued subtraction, is deprived of all number and takes an abiding position (μονήν) and rest." If, after arriving at an unit in this way, we proceed to divide the unit itself into parts, we straightway have multitude again. (3) Some, according to Iamblichus (p. 11, 16), defined it as the "form of forms" (εἰδῶν εἶδος) because it potentially comprehends all forms of number, e.g. it is a polygonal number of any number of sides from three upwards, a solid number in all forms, and so on. (We are forcibly reminded of the latest theories of number as a "Gattung" of "Mengen" or as a "class of classes.") (4) Again an unit, says Iamblichus, is the first, or smallest, in the category of *how many* (ποσόν), the common part or beginning of *how many*. Aristotle defines it as "the indivisible in the (category of) quantity," τὸ κατὰ τὸ ποσὸν ἀδιαίρετον (*Metaph.* 1089 b 35), ποσόν including in Aristotle continuous as well as discrete quantity; hence it is distinguished from a point by the fact that it has not position: "Of the indivisible in the category of, and *quâ*, quantity, that which is every way (indivisible) and destitute of position is called an *unit*, and that which is every way indivisible and has position is a *point*" (*Metaph.* 1016 b 25). (5) In accordance with the last distinction, Aristotle calls the unit "a point without position," στιγμὴ ἄθετος (*Metaph.* 1084 b 26). (6) Lastly, Iamblichus says that the school of Chrysippus defined it in a confused manner (συγκεχυμένως) as "multitude *one* (πλῆθος ἕν)," whereas it is alone contrasted with multitude. On a comparison of these definitions, it would seem that Euclid intended his to be a more *popular* one than those of his predecessors, δημώδης, as Nicomachus called Euclid's definition of an *even number*.

The etymological signification of the word μονάς is supposed by Theon of Smyrna (p. 19, 7—13) to be either (1) that it remains unaltered if it be multiplied by itself any number of times, or (2) that it is separated and *isolated* (μεμονῶσθαι) from the rest of the multitude of numbers. Nicomachus also observes (I. 8, 2) that, while any number is half the sum (1) of the adjacent numbers on each side, (2) of numbers equidistant on each side, the unit is *most solitary* (μονωτάτη) in that it has not a number on each side but only on one side, and it is half of the latter alone, i.e. of 2.

DEFINITION 2.

Ἀριθμὸς δὲ τὸ ἐκ μονάδων συγκείμενον πλῆθος.

The definition of a *number* is again only one out of many that are on record. Nicomachus (I. 7, 1) combines several into one, saying that it is "a defined multitude (πλῆθος ὡρισμένον), or a collection of units (μονάδων σύστημα), or a flow of quantity made up of units" (ποσότητος χύμα ἐκ μονάδων συγκείμενον). Theon, in words almost identical with those attributed by Stobaeus (*Eclogae*, I. 1, 8) to Moderatus, a Pythagorean, says (p. 18, 3—5): "A number is a collection of units, or a progression (προποδισμός) of multitude beginning from an unit and a retrogression (ἀναποδισμός) ceasing at an unit." According to Iamblichus (p. 10) the description "collection of units" (μονάδων σύστημα) was applied to the *how many*, i.e. to number, by Thales, following the Egyptian view (κατὰ τὸ Αἰγυπτιακὸν ἀρέσκον), while it was Eudoxus the Pythagorean who said that a number was "a defined multitude" (πλῆθος ὡρισμένον). Aristotle has a number of definitions which come to the same thing: "limited multitude" (πλῆθος τὸ πεπερασμένον, *Metaph.* 1020 a 13), "multitude" (or "combination") "of units" or "multitude of indivisibles" (*ibid.* 1053 a 30, 1039 a 12, 1085 b 22), "several *ones*" (ἕνα πλείω, *Phys.* III. 7, 207 b 7), "multitude measurable by one" (*Metaph.* 1057 a 3) and "multitude measured and multitude of measures," the "measure" being unity, τὸ ἕν (*ibid.* 1088 a 5).

DEFINITION 3.

Μέρος ἐστὶν ἀριθμὸς ἀριθμοῦ ὁ ἐλάσσων τοῦ μείζονος, ὅταν καταμετρῇ τὸν μείζονα.

By *a part* Euclid means a submultiple, as he does in V. Def. 1, with which definition this one is identical except for the substitution of *number* (ἀριθμός) for *magnitude* (μέγεθος); cf. note on V. Def. 1. Nicomachus uses the word "submultiple" (ὑποπολλαπλάσιος) also. He defines it in a way corresponding to his definition of multiple (see note on Def. 5 below) as follows (I. 18, 2): "The submultiple, which is by nature first in the division of inequality (called) less, is the number which, when compared with a greater, can measure it more times than once so as to fill it exactly (πληρούντως)." Similarly *sub-double* (ὑποδιπλάσιος) is found in Nicomachus meaning *half*, and so on.

DEFINITION 4.

Μέρη δέ, ὅταν μὴ καταμετρῇ.

By the expression *parts* (μέρη, the plural of μέρος) Euclid denotes what we should call a *proper fraction*. That is, *a part* being a submultiple, the rather inconvenient term *parts* means any number of such submultiples making up a fraction less than unity. I have not found the word used in this special sense elsewhere, e.g. in Nicomachus, Theon of Smyrna or Iamblichus, except in one place of Theon (p. 79, 26) where it is used of a proper fraction, of which $\frac{2}{3}$ is an illustration.

DEFINITION 5.

Πολλαπλάσιος δὲ ὁ μείζων τοῦ ἐλάσσονος, ὅταν καταμετρῆται ὑπὸ τοῦ ἐλάσσονος.

The definition of a *multiple* is identical with that in v. Def. 2, except that the masculine of the adjectives is used agreeing with *ἀριθμός* understood instead of the neuter agreeing with *μέγεθος* understood. Nicomachus (I. 18, 1) defines a multiple as being "a species of the greater which is naturally first in order and origin, being the number which, when considered in comparison with another, contains it in itself completely more than once."

DEFINITIONS 6, 7.

6. **Ἄρτιος ἀριθμός ἐστιν ὁ δίχα διαιρούμενος.**

7. **Περισσὸς δὲ ὁ μὴ διαιρούμενος δίχα ἢ [ὁ] μονάδι διαφέρων ἀρτίου ἀριθμοῦ.**

Nicomachus (I. 7, 2) somewhat amplifies these definitions of *even* and *odd* numbers thus. "That is *even* which is capable of being divided into two equal parts without an unit falling in the middle, and that is *odd* which cannot be divided into two equal parts because of the aforesaid intervention (*μεσιτείαν*) of the unit." He adds that this definition is derived "from the popular conception" (*ἐκ τῆς δημώδους ὑπολήψεως*). In contrast to this, he gives (I. 7, 3) the Pythagorean definition, which is, as usual, interesting. "An *even* number is that which admits of being divided, by one and the same operation, into the greatest and the least (parts), greatest in size (*πηλικότητι*) but least in quantity (*ποσότητι*)...while an *odd* number is that which cannot be so treated, but is divided into two unequal parts." That is, as Iamblichus says (p. 12, 2—9), an even number is divided into parts which are the *greatest* possible "parts," namely halves, and into the *fewest* possible, namely two, two being the first "number" or "collection of units." According to another ancient definition quoted by Nicomachus (I. 7, 4), an even number is that which can be divided both into two equal parts and into two unequal parts (except the first one, the number 2, which is only susceptible of division into equals), but, however it is divided, must have its two parts *of the same kind*, i.e. both even or both odd; while an odd number is that which can only be divided into two unequal parts, and those parts always of *different* kinds, i.e. one odd and one even. Lastly, the definition of odd and even "by means of each other" says that an odd number is that which differs by an unit from an even number on both sides of it, and an even number that which differs by an unit from an odd number on each side. This alternative definition of an odd number is the same thing as the second half of Euclid's definition, "the number which differs by an unit from an even number." This evidently pre-Euclidean definition is condemned by Aristotle as unscientific, because odd and even are coordinate, both being *differentiae* of number, so that one should not be defined by means of the other (*Topics* VI. 4, 142 b 7—10).

DEFINITION 8.

Ἀρτιάκις ἄρτιος ἀριθμός ἐστιν ὁ ὑπὸ ἀρτίου ἀριθμοῦ μετρούμενος κατὰ ἄρτιον ἀριθμόν.

Euclid's definition of an *even-times even* number differs from that given by the later writers, Nicomachus, Theon of Smyrna and Iamblichus; and the inconvenience of it is shown when we come to IX. 34, where it is proved that a certain sort of number is *both* "even-times even" and "even-times odd." According to the more precise classification of the three other authorities, the "even-times even" and the "even-times odd" are mutually exclusive and are two of three subdivisions into which even numbers fall. Of these three subdivisions the "even-times even" and the "even-times odd" form the extremes, and the "odd-times even" is as it were intermediate, showing the character of both extremes (cf. note on the following definition). The *even-times even* is then the number which has its halves even, the halves of the halves even, and so on, until unity is reached. In short the *even-times even* number is always of the form 2^n. Hence Iamblichus (pp. 20, 21) says Euclid's definition of it as that which is measured by an even number an even number of times is erroneous. In support of this he quotes the number 24 which is four times 6, or six times 4, but yet is not "even-times even" according to Euclid himself (οὐδὲ κατ' αὐτόν), by which he must apparently mean that 24 is also 8 times 3, which does not satisfy Euclid's definition. There can however be no doubt that Euclid meant what he said in his definition as we have it; otherwise IX. 32, which proves that a number of the form 2^n is *even-times even only*, would be quite superfluous and a mere repetition of the definition, while, as already stated, IX. 34 clearly indicates Euclid's view that a number might at the same time be both even-times even and even-times odd. Hence the μόνως which some editor of the commentary of Philoponus on Nicomachus found in some copies, making the definition say that the even-times even number is *only* measured by even numbers an even number of times, is evidently an interpolation by some one who wished to reconcile Euclid's definition with the Pythagorean (cf. Heiberg, *Euklid-studien*, p. 200).

A consequential characteristic of the series of even-times even numbers noted by Nicomachus brings in a curious use of the word δύναμις (generally *power* in the sense of square, or square root). He says (I. 8, 6—7) that any part, i.e. any submultiple, of an even-times even number is called by an even-times even designation, while it also has an even-times even *value* (it is ἀρτιάκις ἀρτιοδύναμον) when expressed as so many actual units. That is, the $\frac{1}{2^m}$th part of 2^n (where m is less than n) is called after the even-times even number 2^m, while its actual *value* (δύναμις) in units is 2^{n-m}, which is also an even-times even number. Thus all the parts, or submultiples, of even-times even numbers, as well as the even-times even numbers themselves, are connected with one kind of number only, the *even*.

DEFINITION 9.

Ἀρτιάκις δὲ περισσός ἐστιν ὁ ὑπὸ ἀρτίου ἀριθμοῦ μετρούμενος κατὰ περισσὸν ἀριθμόν.

Euclid uses the term *even-times odd* (ἀρτιάκις περισσός), whereas Nicomachus and the others make it one word, *even-odd* (ἀρτιοπέριττος). According to the stricter definition given by the latter (I. 9, 1), the *even-odd* number is related to the *even-times even* as the other extreme. It is such a number as, when once halved, leaves as quotient an odd number; that is, it is of the form $2(2m+1)$. Nicomachus sets the even-odd numbers out as follows,

6, 10, 14, 18, 22, 26, 30, etc.

In this case, as Nicomachus observes, any part, or submultiple, is called by a name *not* corresponding in kind to its actual value (δύναμις) in units. Thus, in the case of 18, the $\frac{1}{2}$ part is called after the even number 2, but its *value* is the odd number 9, and the $\frac{1}{3}$rd part is called after the odd number 3, while its *value* is the even number 6, and so on.

The third class of even numbers according to the strict subdivision is the *odd-even* (περισσάρτιος). Numbers are of this class when they can be halved twice or more times successively, but the quotient left when they can no longer be halved is an odd number and not unity. They are therefore of the form $2^{n+1}(2m+1)$, where n, m are integers. They are, so to say, intermediate between, or a mixture of, the extreme classes *even-times even* and *even-odd*, for the following reasons. (1) Their subdivision by 2 proceeds for some way like that of the even-times even, but ends in the way that the division of the even-odd by 2 ends. (2) The numbers after which submultiples are called and their *value* (δύναμις) in units may be both of one kind, i.e. both odd or both even (as in the case of the even-times even), or again may be one odd and one even as in the case of the even-odd. For example 24 is an odd-even number; the $\frac{1}{4}$th, $\frac{1}{12}$th, $\frac{1}{6}$th or $\frac{1}{2}$ parts of it are even, but the $\frac{1}{3}$rd part of it, or 8, is even, and the $\frac{1}{8}$th part of it, or 3, is odd. (3) Nicomachus shows (I. 10, 6—9) how to form all the numbers of the odd-even class. Set out two lines (*a*) of odd numbers beginning with 3, (*b*) of even-times even numbers beginning with 4, thus:

(*a*) 3, 5, 7, 9, 11, 13, 15 etc.
(*b*) 4, 8, 16, 32, 64, 128, 256 etc.

Now multiply each of the first numbers into each of the second row. Let the products of one of the first into all the second set make horizontal rows; we then get the rows

12,	24,	48,	96,	192,	384,	768	etc.
20,	40,	80,	160,	320,	640,	1280	etc.
28,	56,	112,	224,	448,	896,	1792	etc.
36,	72,	144,	288,	576,	1152,	2304	etc.

and so on.

Now, says Nicomachus, you will be surprised to see (φανήσεταί σοι θαυμαστῶς) that (*a*) the *vertical* rows have the property of the *even-odd* series, 6, 10, 14, 18, 22 etc., viz. that, if an odd number of successive numbers be taken, the middle number is half the sum of the extremes, and if an even number, the two middle numbers together are equal to the sum of the extremes, (*b*) the *horizontal* rows have the property of the *even-times even* series 4, 8, 16 etc., viz. that the product of the extremes of any number of successive terms is equal, if their number be odd, to the square of the middle term, or, if their number be even, to the product of the two middle terms.

Let us now return to Euclid. His 9th definition states that an *even-times odd* number is a number which, when divided by an even number, gives an odd number as quotient. Following this definition in our text comes a 10th definition which defines an *odd-times even* number; this is stated to be a number which, when divided by an odd number, gives an even number as quotient. According to these definitions any *even-times odd* number would also be *odd-times even*, and, from the fact that Iamblichus notes this, we may fairly conclude that he found Def. 10 as well as Def. 9 in the text of Euclid which he used. But, if both definitions are genuine, the enunciations of IX. 33 and IX. 34 as we have them present difficulties. IX. 33 says that "If a number have its half odd, it is even-times odd *only*"; but, on the assumption that both definitions are genuine, this would not be true, for the number would be *odd-times even* as well. IX. 34 says that "If a number neither be one of those which are continually doubled from 2, nor have its half odd, it is both even-times even and even-times odd." The term *odd-times even* (περισσάκις ἄρτιος) not occurring in these propositions, nor anywhere else after the definition, that definition becomes superfluous. Iamblichus however (p. 24, 7—14) quotes these enunciations differently. In the first he has instead of "even-times odd only" the words "*both even-times odd and odd-times even*"; and, in the second, for "both even-times even and even-times odd" he has "is both even-times even and at the same time even-times odd and odd-times even." In both cases therefore "odd-times even" is added to the enunciation as Iamblichus had it; the words cannot have been added by Iamblichus himself because he himself does not use the term *odd-times even*, but the one word *odd-even* (περισσάρτιος). In order to get over the difficulties involved by Def. 10 and these differences of reading we have practically to choose between (1) accepting Iamblichus' reading in all three places and (2) adhering to the reading of our MSS. in IX. 33, 34 and rejecting Def. 10 altogether as an interpolation. Now the readings of our text of IX. 33, 34 are those of the Vatican MS. and the Theonine MSS. as well; hence they must go back to a time before Theon, and must therefore be almost as old as those of Iamblichus. Heiberg considers it improbable that Euclid would wish to maintain a pointless distinction between *even-times odd* and *odd-times even*, and on the whole concludes that Def. 10 was first interpolated by some ignorant person who did not notice the difference between the Euclidean and Pythagorean classification, but merely noticed the absence of a definition of *odd-times even* and fabricated one as a companion to the other. When this was done, it would be easy to see that the statement in IX. 33 that the number referred to is "even-times odd *only*" was not strictly true, and that the addition of the words "and odd-times even" was necessary in IX. 33 and IX. 34 as well.

Definition 10.

Περισσάκις δὲ περισσὸς ἀριθμός ἐστιν ὁ ὑπὸ περισσοῦ ἀριθμοῦ μετρούμενος κατὰ περισσὸν ἀριθμόν.

The *odd-times odd* number is not defined as such by Nicomachus and Iamblichus; for them these numbers would apparently belong to the *composite* subdivision of *odd* numbers. Theon of Smyrna on the other hand says (p. 23, 21) that *odd-times odd* was one of the names applied to *prime* numbers (excluding 2), for these have two odd factors, namely 1 and the number itself. This is certainly a curious use of the term.

Definition 11.

Πρῶτος ἀριθμός ἐστιν ὁ μονάδι μόνῃ μετρούμενος.

A *prime* number (πρῶτος ἀριθμός) is called by Nicomachus, Theon, and Iamblichus a "prime *and incomposite* (ἀσύνθετος) number." Theon (p. 23, 9) defines it practically as Euclid does, viz. as a number "measured by no number, but by an unit only." Aristotle too says that a prime number is not measured by any number (*Anal. post.* II. 13, 96 a 36), an unit not being a number (*Metaph.* 1088 a 6), but only the beginning of number (Theon of Smyrna says the same thing, p. 24, 23). According to Nicomachus (I. 11, 2) the prime number is a subdivision, not of numbers, but of *odd* numbers; it is "an odd number which admits of no other part except that which is called after its own name (παρώνυμον ἑαυτῷ)." The prime numbers are 3, 5, 7 etc., and there is no submultiple of 3 except $\frac{1}{3}$rd, no submultiple of 11 except $\frac{1}{11}$th, and so on. In all these cases the only submultiple is an unit. According to Nicomachus 3 is the first prime number, whereas Aristotle (*Topics* VIII. 2, 157 a 39) regards 2 as a prime number: "as the dyad is the only even number which is prime," showing that this divergence from the Pythagorean doctrine was earlier than Euclid. The number 2 also satisfies Euclid's definition of a prime number. Iamblichus (p. 30, 27 sqq.) makes this the ground of another attack upon Euclid. His argument (the text of which, however, leaves much to be desired) appears to be that 2 is the *only* even number which has no other part except an unit, while the subdivisions of the even, as previously explained by him (the *even-times even*, the *even-odd*, and *odd-even*), all exclude primeness, and he has previously explained that 2 is *potentially* even-odd, being obtained by multiplying by 2 the *potentially* odd, i.e. the unit; hence 2 is regarded by him as bound up with the subdivisions of even, which exclude primeness. Theon seems to hold the same view as regards 2, but supports it by an apparent circle. A prime number, he says (p. 23, 14—23), is also called *odd-times odd*; therefore only odd numbers are prime and incomposite. Even numbers are not measured by the unit alone, except 2, which therefore (p. 24, 7) is odd-*like* (περισσοειδής) without being prime.

A variety of other names were applied to prime numbers. We have already noted the curious designation of them as *odd-times odd*. According to Iamblichus (p. 27, 3—5) some called them *euthymetric* (εὐθυμετρικός), and Thymaridas *rectilinear* (εὐθυγραμμικός), the ground being that they can only be set out in one dimension with no breadth (ἀπλατὴς γὰρ ἐν τῇ ἐκθέσει ἐφ' ἓν μόνον διιστάμενος). The same aspect of a prime number is also expressed by Aristotle, who (*Metaph.* 1020 b 3) contrasts the composite number with that which is only in one dimension (μόνον ἐφ' ἓν ὤν). Theon of Smyrna (p. 23, 12) gives γραμμικός (*linear*) as the alternative name instead of εὐθυγραμμικός. In either case, to make the word a proper description of a prime number we have to understand the word *only*; a prime number is that which is *linear*, or *rectilinear*, *only*. For Nicomachus, who uses the form *linear*, expressly says (II. 13, 6) that *all* numbers are so, i.e. all can be represented as linear by dots to the required amount placed in a line.

A prime number was called *prime* or *first*, according to Nicomachus (I. 11, 3), because it can only be arrived at by putting together a certain number of units, and the unit is the beginning of number (cf. Aristotle's second sense of πρῶτος "as not being composed of *numbers*," ὡς μὴ συγκεῖσθαι ἐξ ἀριθμῶν, *Anal. Post.* II. 13, 96 a 37), and also, according to Iamblichus, because there is no number before it, being a collection of units (μονάδων σύστημα), of which it is a multiple, and it appears *first* as a basis for other numbers to be multiples of.

DEFINITION 12.

Πρῶτοι πρὸς ἀλλήλους ἀριθμοί εἰσιν οἱ μονάδι μόνῃ μετρούμενοι κοινῷ μέτρῳ.

By way of further emphasising the distinction between "prime" and "prime to one another," Theon of Smyrna (p. 23, 6—8) calls the former "prime *absolutely*" (ἁπλῶς), and the latter "prime to one another and *not* *absolutely*" or "*not in themselves*" (οὐ καθ' αὑτούς). The latter (p. 24, 8—10) are "measured by the unit [sc. only] as common measure, even though, taken by themselves (ὡς πρὸς ἑαυτούς), they be measured by some other numbers." From Theon's illustrations it is clear that with him as with Euclid a number prime to another may be even as well as odd. In Nicomachus (I. 11, 1) and Iamblichus (p. 26, 19), on the other hand, the number which is "in itself secondary (δεύτερος) and composite (σύνθετος), but in relation to another prime and incomposite," is a subdivision of *odd*. I shall call more particular attention to this difference of classification when we have reached the definitions of "composite" and "composite to one another"; for the present it is to be noted that Nicomachus (I. 13, 1) defines a number prime *to another* after the same manner as the absolutely prime; it is a number which "is measured not only by the unit as the common measure but also by some other measure, and for this reason can also admit of a part or parts called by a different name besides that called by the same name (as itself), but, when examined in comparison with another number of similar character, is found not to be capable of being measured by a common measure in relation to the other, nor to have the same part, called by the same name as (any of) those simply (ἁπλῶς) contained in the other; e.g. 9 in relation to 25, for each of these is in itself secondary and composite, but, in comparison with one another, they have an unit alone as a common measure and no part is called by the same name in both, but the *third* in one is not in the other, nor is the *fifth* in the other found in the first."

DEFINITION 13.

Σύνθετος ἀριθμός ἐστιν ὁ ἀριθμῷ τινι μετρούμενος.

Euclid's definition of *composite* is again the same as Theon's definition of numbers "composite in relation to themselves," which (p. 24, 16) are "numbers measured by any less number," the unit being, as usual, not regarded as a number. Theon proceeds to say that "of composite numbers they call those which are contained by two numbers *plane*, as being investigated in two dimensions and, as it were, contained by a length and a breadth, while (they call) those (which are contained) by three (numbers) *solid*, as having the third dimension added to them." To a similar effect is the remark of Aristotle (*Metaph.* 1020 b 3) that certain numbers are "composite and are not only in one dimension but such as the plane and the solid (figure) are representations of (μίμημα), these numbers being so many times so many (ποσάκις ποσοί), or so many times so many times so many (ποσάκις ποσάκις ποσοί) respectively." These subdivisions of composite numbers are, of course, the subject of Euclid's definitions 17, 18 respectively. Euclid's composite numbers may be either even or odd, like those of Theon, who gives 6 as an instance, 6 being measured by both 2 and 3.

DEFINITION 14.

Σύνθετοι δὲ πρὸς ἀλλήλους ἀριθμοί εἰσιν οἱ ἀριθμῷ τινι μετρούμενοι κοινῷ μέτρῳ.

Theon (p. 24, 18), like Euclid, defines numbers *composite to one another* as "those which are measured by any common measure whatever" (excluding unity, as usual). Theon instances 8 and 6, with 2 as common measure, and 6 and 9, with 3 as common measure.

As hinted above, there is a great difference between Euclid's classification of prime and composite numbers, and of numbers prime and composite to one another, and the classification found in Nicomachus (I. 11—13) and Iamblichus. According to the latter, all these kinds of numbers are subdivisions of the class of *odd* numbers only. As the class of *even* numbers is divided into three kinds, (1) the even-times even, (2) the even-odd, which form the extremes, and (3) the odd-even, which is, as it were, intermediate to the other two, so the class of *odd* numbers is divided into three, of which the third is again a mean between two extremes. The three are:

(1) the *prime and incomposite*, which is like Euclid's prime number except that it excludes 2;

(2) the *secondary and composite*, which is "odd because it is a distinct part of one and the same genus (διὰ τὸ ἐξ ἑνὸς καὶ τοῦ αὐτοῦ γένους διακεκρίσθαι) but has in it nothing of the nature of a first principle (ἀρχοειδές); for it arises from adding some other number (to itself), so that, besides having a part called by the same name as itself, it possesses a part or parts called by another name." Nicomachus cites 9, 15, 21, 25, 27, 33, 35, 39. It is made clear that not only must the factors be both odd, but they must all be prime numbers. This is obviously a very inconvenient restriction of the use of the word *composite*, a word of general signification.

(3) is that which is "*secondary and composite in itself but prime and incomposite to another*." The actual words in which this is defined have been given above in the note on Def. 12. Here again all the factors must be odd and prime.

Besides the inconvenience of restricting the term *composite* to *odd* numbers which are composite, there is in this classification the further serious defect, pointed out by Nesselmann (*Die Algebra der Griechen*, 1842, p. 194), that subdivisions (2) and (3) overlap, subdivision (2) including the whole of subdivision (3). The origin of this confusion is no doubt to be found in Nicomachus' perverse anxiety to be symmetrical; by hook or by crook he must divide *odd* numbers into three kinds as he had divided the *even*. Iamblichus (p. 28, 13) carries his desire to be logical so far as to point out why there cannot be a fourth kind of number contrary in character to (3), namely a number which should be "prime and incomposite in itself, but secondary and composite to another"!

DEFINITION 15.

Ἀριθμὸς ἀριθμὸν πολλαπλασιάζειν λέγεται, ὅταν, ὅσαι εἰσὶν ἐν αὐτῷ μονάδες, τοσαυτάκις συντεθῇ ὁ πολλαπλασιαζόμενος, καὶ γένηταί τις.

This is the well known primary definition of multiplication as an abbreviation of addition.

DEFINITION 16.

Ὅταν δὲ δύο ἀριθμοὶ πολλαπλασιάσαντες ἀλλήλους ποιῶσί τινα, ὁ γενόμενος ἐπίπεδος καλεῖται, πλευραὶ δὲ αὐτοῦ οἱ πολλαπλασιάσαντες ἀλλήλους ἀριθμοί.

The words *plane* and *solid* applied to numbers are of course adapted from their use with reference to geometrical figures. A number is therefore called *linear* (γραμμικός) when it is regarded as in one dimension, as being a *length* (μῆκος). When it takes another dimension in addition, namely *breadth* (πλάτος), it is in two dimensions and becomes *plane* (ἐπίπεδος). The distinction between a *plane* and a *plane number* is marked by the use of the neuter in the former case, and the masculine, agreeing with ἀριθμός, in the latter case. So with a *square* and a *square number*, and so on. The most obvious form of a plane number is clearly that corresponding to a rectangle in geometry; the number is the product of two linear numbers regarded as *sides* (πλευραί) forming the length and breadth respectively. Such a number is, as Aristotle says, "so many times so many," and a plane is its counterpart (μίμημα). So Plato, in the *Theaetetus* (147 E—148 B), says: "We divided all numbers into two kinds, (1) that which can be expressed as equal multiplied by equal (τὸν δυνάμενον ἴσον ἰσάκις γίγνεσθαι), and which, likening its form to the square, we called *square* and equilateral; (2) that which is intermediate, and includes 3 and 5 and every number which cannot be expressed as equal multiplied by equal, but is either less times more or more times less, being always contained by a greater and a less side, which number we likened to the oblong figure (προμήκει σχήματι) and called an *oblong* number.... Such *lines* therefore as *square the equilateral and plane number* [i.e. which can form a plane number with equal sides, or a square] we defined as *length* (μῆκος); but such as square the oblong (here ἑτερομήκης) [i.e. the square of which is equal to the oblong] we called *roots* (δυνάμεις) as not being commensurable with the others in length, but only in the plane areas (ἐπιπέδοις), to which the squares on them are equal (ἃ δύνανται)." This passage seems to make it clear that Plato would have represented numbers as Euclid does, by straight lines proportional in length to the numbers they represent (so far as practicable); for, since 3 and 5 are with Plato oblong numbers, and *lines* with him represent the sides of oblong numbers (since a line represents the "root," the square on which is equal to the oblong), it follows that the *unit* representing the smaller side must have been represented as a line, and 3, the larger side, as a line of three times the length. But there is another possible way of representing numbers, not by lines of a certain length, but by *points* disposed in various ways, in straight lines or otherwise. Iamblichus tells us (p. 56, 27) that "in old days they represented the quantuplicities of number in a more natural way (φυσικώτερον) by splitting them up into units, and not, as in our day, by symbols" (συμβολικῶς). Aristotle too (*Metaph.* 1092 b 10) mentions one Eurytus as having settled what number belonged to what, such a number to a man, such a number to a horse, and so on, "copying their shapes" (reading τούτων, with Zeller) "*with pebbles* (ταῖς ψήφοις), *just as those do who arrange numbers in the forms of triangles or squares.*" We accordingly find numbers represented in Nicomachus and Theon of Smyrna by a number of α's ranged like points according to geometrical figures. According to this system, any number could be represented by points in a straight line, in which case, says Iamblichus (p. 56, 26), we shall call it rectilinear because it is *without breadth* and only advances in length (ἁπλατῶς ἐπὶ μόνον τὸ μῆκος πρόεισιν). The prime number was called by Thymaridas rectilinear *par excellence*, because it was without breadth and in one dimension *only* (ἐφ' ἓν μόνον διστάμενος). By this must be meant the impossibility of representing, say, 3 as a plane number, in Plato's sense, i.e. as a product of two numbers corresponding to a rectangle in geometry; and this view would appear to rest simply upon the representation of a number by *points*, as distinct from lines. Three dots in a straight line would have *no* breadth; and if breadth were introduced in the sense of producing a rectangle, i.e. by placing the same number of dots in a second line below the first line, the first *plane* number would be 4, and 3 would not be a plane number at all, as Plato says it is. It seems therefore to have been the alternative representation of a number by points, and not lines, which gave rise to the different view of a plane number which we find in Nicomachus and the rest. By means of separate points we can represent numbers in geometrical forms other than rectangles and squares. One dot with two others symmetrically arranged below it shows a *triangle*, which is a figure *in two dimensions* as much as a rectangle or parallelogram is. Similarly we can arrange certain numbers in the form of regular *pentagons* or other polygons. According therefore to this mode of representation, 3 is the first *plane* number, being a *triangular* number. The method of formation of triangular, square, pentagonal and other polygonal numbers is minutely described in Nicomachus (II. 8—11), who distinguishes the separate series of *gnomons* belonging to each, i.e. gives the law determining the number which has to be added to a polygonal number with n in a side, in order to make it into a number of the same form but with $n + 1$ in a side (the addend being of course the gnomon). Thus the gnomonic series for triangular numbers is 1, 2, 3, 4, 5...; that for squares 1, 3, 5, 7...; that for pentagonal numbers 1, 4, 7, 10..., and so on. The subject need not detain us longer here, as we are at present only concerned with the different views of what constitutes a *plane* number.

Of *plane* numbers in the Platonic and Euclidean sense we have seen that Plato recognises *two* kinds, the *square* and the *oblong* (προμήκης or ἑτερομήκης). Here again Euclid's successors, at all events, subdivided the class more elaborately. Nicomachus, Theon of Smyrna, and Iamblichus divide *plane* numbers with unequal *sides* into (1) ἑτερομήκεις, the nearest thing to squares, viz. numbers in which the greater side exceeds the less side by 1 only, or numbers of the form $n(n + 1)$, e.g. 1.2, 2.3, 3.4, etc. (according to Nicomachus), and (2) προμήκεις, or those whose sides differ by 2 or more, i.e. are of the form $n(n + m)$, where m is not less than 2 (Nicomachus illustrates by 2.4, 3.6, etc.). Theon of Smyrna (p. 30, 8—14) makes προμήκεις include ἑτερομήκεις, saying that their sides may differ by 1 or more; he also speaks of *parallelogram*-numbers as those which have one side different from the other by 2 or more; I do not find this latter term in Nicomachus or Iamblichus, and indeed it seems superfluous, as *parallelogram* is here only another name for oblong. Iamblichus (p. 74, 23 sqq.), always critical of Euclid, attacks him again here for confusing the subject by supposing that the ἑτερομήκης number is the product of *any* two different numbers multiplied together, and by not distinguishing the oblong (προμήκης) from it: "for his definition declares the same number to be square and also ἑτερομήκης, as for example 36, 16 and many others: which would be equivalent to the odd number being the same thing as the even." No importance need be attached to this exaggerated statement; it is in any case merely a matter of words, and it is curious that Euclid does not in fact use the word ἑτερομήκης of *numbers* at all, but only of geometrical oblong figures as opposed to squares, so that Iamblichus can apparently only have inferred that he used it in an unorthodox manner from the *geometrical* use of the term in the definitions of Book I. and from the fact that he does not give the two subdivisions of *plane numbers* which are not square, but seems only to divide *plane numbers* into square and not-square. The argument that ἑτερομήκεις numbers are a *natural*, and therefore essential, subdivision Iamblichus appears to found on the method of successive addition by which they can be evolved; as square numbers are obtained by successively adding *odd* numbers as gnomons, so ἑτερομήκεις are obtained by adding *even* numbers as gnomons. Thus $1 \cdot 2 = 2$, $2 \cdot 3 = 2 + 4$, $3 \cdot 4 = 2 + 4 + 6$, and so on.

DEFINITION 17.

Ὅταν δὲ τρεῖς ἀριθμοὶ πολλαπλασιάσαντες ἀλλήλους ποιῶσί τινα, ὁ γενόμενος στερεός ἐστιν, πλευραὶ δὲ αὐτοῦ οἱ πολλαπλασιάσαντες ἀλλήλους ἀριθμοί.

What has been said of the two apparently different ways of regarding a *plane* number seems to apply equally, *mutatis mutandis*, to the definitions of a *solid* number. Aristotle regards it as a number which is so many times so many times so many (ποσάκις ποσάκις ποσόν). Plato finishes the passage about lines which represent the sides of *square* numbers and lines which are *roots* (δυνάμεις), i.e. the squares on which are equal to the rectangle representing a number which is oblong and not square, by adding the words, "And another similar property belongs to solids" (καὶ περὶ τὰ στερεὰ ἄλλο τοιοῦτον). That is, apparently, there would be a corresponding term to *root* (δύναμις)—practically representing a surd—to denote the side of a cube equal to a parallelepiped representing a solid number which is the product of three factors but not a cube. Such is a solid number when numbers are represented by *straight lines*: it corresponds in general to a parallelepiped and, when all the factors are equal, to a cube.

But again, if numbers be represented by *points*, we may have solid numbers (i.e. numbers in three dimensions) in the form of *pyramids* as well. The first number of this kind is 4, since we may have three points forming an equilateral triangle in one plane and a fourth point placed in another plane. The length of the sides can be increased by 1 successively; and we can have a series of pyramidal numbers, with triangles, squares or polygons as bases, made up of layers of triangles, squares or similar polygons respectively, each of which layers has one less in the side than the layer below it, until the top of the pyramid is reached, which of course is one point representing unity. Nicomachus (II. 13—16), Theon of Smyrna (p. 41—2), and Iamblichus (p. 95, 15 sqq.), all give the different kinds of *pyramidal* solid numbers in addition to the other kinds.

These three writers make the following further distinctions between solid numbers which are the product of three factors.

1. First there is the equal by equal by equal (ἰσάκις ἰσάκις ἴσος), which is, of course, the cube.

2. The other extreme is the unequal by unequal by unequal (ἀνισάκις ἀνισάκις ἄνισος), or that in which all the dimensions are different, e.g. the product of 2, 3, 4 or 2, 4, 8 or 3, 5, 12. These were, according to Nicomachus (II. 16), called *scalene*, while some called them σφηνίσκοι (*wedge-shaped*), others σφηκίσκοι (from σφήξ, a *wasp*), and others βωμίσκοι (*altar-shaped*). Theon appears to use the last term only, while Iamblichus of course gives all three names.

3. Intermediate to these, as it were, come the numbers "whose *planes* form ἑτερομήκεις numbers" (i.e. numbers of the form $n(n+1)$). These, says Nicomachus, are called *parallelepipedal*.

Lastly come two classes of such numbers each of which has two equal dimensions but not more.

4. If the third dimension is less than the others, the number is *equal by equal by less* (ἰσάκις ἴσος ἐλαττονάκις) and is called a *plinth* (πλινθίς), e.g. 8 . 8 . 3.

5. If the third dimension is greater than the others, the number is *equal by equal by greater* (ἰσάκις ἴσος μειζονάκις) and is called a *beam* (δοκίς), e.g. 3 . 3 . 7. Another name for this latter kind of number (according to Iamblichus) was στηλίς (diminutive of στήλη).

Lastly, in connexion with pyramidal numbers, Nicomachus (II. 14, 5) distinguishes numbers corresponding to *frusta* of pyramids. These are *truncated* (κόλουροι), *twice-truncated* (δικόλουροι), *thrice-truncated* (τρικόλουροι) pyramids, and so on, the term being used mostly in theoretic treatises (ἐν συγγράμμασι μάλιστα τοῖς θεωρηματικοῖς). The *truncated* pyramid was formed by cutting off the point forming the vertex. The *twice-truncated* was that which lacked the vertex and the next plane, and so on. Theon of Smyrna (p. 42, 4) only mentions the *truncated* pyramid as "that with its vertex cut off" (ἡ τὴν κορυφὴν ἀποτετμημένη), saying that some also called it a trapezium, after the similitude of a plane trapezium formed by cutting the top off a triangle by a straight line parallel to the base.

DEFINITION 18.

Τετράγωνος ἀριθμός ἐστιν ὁ ἰσάκις ἴσος ἢ [ὁ] ὑπὸ δύο ἴσων ἀριθμῶν περιεχόμενος.

A particular kind of square distinguished by Nicomachus and the rest was the square number which ended (in the decimal notation) with the same number as its side, e.g. 1, 25, 36, which are the squares of 1, 5 and 6. These square numbers were called *cyclic* (κυκλικοί) on the analogy of circles in geometry which return again to the point from which they started.

DEFINITION 19.

Κύβος δὲ ὁ ἰσάκις ἴσος ἰσάκις ἢ [ὁ] ὑπὸ τριῶν ἴσων ἀριθμῶν περιεχόμενος.

Similarly cube numbers which ended with the same number as their sides, *and the squares of those sides also*, were called *spherical* (σφαιρικοί) or *recurrent* (ἀποκαταστατικοί). One might have expected that the term *spherical* would be applicable also to the cubes of numbers which ended with the same digit as the *side* but not necessarily with the same digit as the *square* of the side also. E.g. the cube of 4, i.e. 64, ends with the same digit as 4, but not with the same digit as 16. But apparently 64 was not called a spherical number, the only instances given by Nicomachus and the rest being those cubed from numbers ending with 5 or 6, which end with the same digit if *squared*. A *spherical* number is in fact derived from a *circular* number only, and that by adding another equal dimension. Obviously, as Nesselmann says, the names *cyclic* and *spherical* applied to numbers appeal to an entirely different principle from that on which the figured numbers so far dealt with were formed.

DEFINITION 20.

Ἀριθμοὶ ἀνάλογόν εἰσιν, ὅταν ὁ πρῶτος τοῦ δευτέρου καὶ ὁ τρίτος τοῦ τετάρτου ἰσάκις ᾖ πολλαπλάσιος ἢ τὸ αὐτὸ μέρος ἢ τὰ αὐτὰ μέρη ὦσιν.

Euclid does not give in this Book any definition of ratio, doubtless because it could only be the same as that given at the beginning of Book V., with numbers substituted for "homogeneous magnitudes" and "in respect of *size*" (πηλικότητα) omitted or altered. We do not find that Nicomachus and the rest give any substantially different definition of a *ratio* between numbers. Theon of Smyrna says, in fact (p. 73, 16), that "ratio in the sense of proportion (λόγος ὁ κατ' ἀνάλογον) is a sort of relation of two homogeneous terms to one another, as for example, double, triple." Similarly Nicomachus says (II. 21, 3) that "a ratio is a relation of two terms to one another," the word for "relation" being in both cases the same as Euclid's (σχέσις). Theon of Smyrna goes on to classify ratios as greater, less, or equal, i.e. as ratios of greater inequality, less inequality, or equality, and then to specify certain arithmetical ratios which had special names, for which he quotes the authority of Adrastus. The names were πολλαπλάσιος, ἐπιμόριος, ἐπιμερής, πολλαπλασιεπιμόριος, πολλαπλασιεπιμερής (the first of which is, of course, a multiple, while the rest are the equivalent of certain types of improper fractions as we should call them), and the reciprocals of each of these described by prefixing ὑπό or *sub*. After describing these particular classes of arithmetical ratios, Theon goes on to say that numbers still have ratios to one another even if they are different from all those previously described. We need not therefore concern ourselves with the various types; it is sufficient to observe that any ratio between numbers can be expressed in the manner indicated in Euclid's definition of arithmetical proportion, for the greater is, in relation to the less, either one or a combination of more than one of the three things, (1) a multiple, (2) a submultiple, (3) a proper fraction.

It is when we come to the definition of *proportion* that we begin to find differences between Euclid, Nicomachus, Theon and Iamblichus. "Proportion," says Theon (p. 82, 6), "is similarity or sameness of more ratios than one," which is of course unobjectionable if it is previously understood what a *ratio* is; but confusion was brought in by those (like Thrasyllus) who said that there were *three proportions* (ἀναλογίαι), the arithmetic, geometric, and harmonic, where of course the reference is to arithmetic, geometric and harmonic *means* (μεσότητες). Hence it was necessary to explain, as Adrastus did (Theon, p. 106, 15), that of the several *means* "the geometric was called both proportion *par excellence* and primary...though the other means were also commonly called *proportions* by some writers." Accordingly we have Nicomachus trying to extend the term "proportion" to cover the various *means* as well as a proportion in three or four terms in the ordinary sense. He says (II. 21, 2): "Proportion, *par excellence* (κυρίως), is the bringing together (σύλληψις) to the same (point) of two or more *ratios*; or, more generally, (the bringing together) of two or more *relations* (σχέσεων), even though they be subjected not to the same *ratio* but to a difference or some other (law)." Iamblichus keeps the senses of the word more distinct. He says, like Theon, that "proportion is similarity or sameness of several ratios" (p. 98, 14), and that "it is to be premised that it was the geometrical (proportion) which the ancients called proportion *par excellence*, though it is now common to apply the name generally to all the remaining means as well" (p. 100, 15). Pappus remarks (III. p. 70, 17), "A mean differs from a proportion in this respect that, if anything is a proportion, it is also a mean, but not conversely. For there are three means, of which one is arithmetic, one geometric and one harmonic." The last remark implies plainly enough that there is only one *proportion* (ἀναλογία) in the proper sense. So, too, says Iamblichus in another place (p. 104, 19): "the second, the geometric, mean has been called *proportion par excellence* because the terms contain the same ratio, being separated according to the same proportion (ἀνὰ τὸν αὐτὸν λόγον διεστῶτες)." The natural conclusion is that of Nesselmann, that originally the geometric proportion was called ἀναλογία, the others, the arithmetic, the harmonic, etc., *means*; but later usage had obliterated the distinction.

Of proportions in the ancient and Euclidean sense Theon (p. 82, 10) distinguished the *continuous* (συνεχής) and the *separated* (διῃρημένη), using the same terms as Aristotle (*Eth. Nic.* 1131 a 32). The meaning is of course clear: in the *continuous* proportion the consequent of one ratio is the antecedent of the next; in the *separated* proportion this is not so. Nicomachus (II. 21, 5—6) uses the words *connected* (συνημμένη) and *disjoined* (διεζευγμένη) respectively. Euclid regularly speaks of numbers in continuous proportion as "proportional in order, or successively" (ἑξῆς ἀνάλογον).

DEFINITION 21.

Ὅμοιοι ἐπίπεδοι καὶ στερεοὶ ἀριθμοί εἰσιν οἱ ἀνάλογον ἔχοντες τὰς πλευράς.

Theon of Smyrna remarks (p. 36, 12) that, among plane numbers, *all* squares are similar, while of ἑτερομήκεις those are similar "whose sides, that is, the numbers containing them, are proportional." Here ἑτερομήκης must evidently be used, not in the sense of a number of the form $n(n+1)$, but as synonymous with προμήκης, *any* oblong number; so that on this occasion Theon follows the terminology of Plato and (according to Iamblichus) of Euclid. Obviously, if the strict sense of ἑτερομήκης is adhered to, no two numbers of that form can be similar unless they are also *equal*. We may compare Iamblichus' elaborate contrast of the square and the ἑτερομήκης. Since the two sides of the square are equal, a square number might, as he says (p. 82, 9), be fitly called ἰδιομήκης (Nicomachus uses ταὐτομήκης) in contrast to ἑτερομήκης; and the ancients, according to him, called square numbers "the same" and "similar" (ταὐτούς τε καὶ ὁμοίους), but ἑτερομήκεις numbers "dissimilar and other" (ἀνομοίους καὶ θατέρους).

With regard to solid numbers, Theon remarks in like manner (p. 37, 2) that *all* cube numbers are similar, while of the others those are similar whose sides are proportional, i.e. in which, as length is to length, so is breadth to breadth and height to height.

DEFINITION 22.

Τέλειος ἀριθμός ἐστιν ὁ τοῖς ἑαυτοῦ μέρεσιν ἴσος ὤν.

Theon of Smyrna (p. 45, 9 sqq.) and Nicomachus (I. 16) both give the same definition of a *perfect* number, as well as the law of formation of such numbers which Euclid proves in the later proposition, IX. 36. They add however definitions of two other kinds of numbers in contrast with it, (1) the *over-perfect* (ὑπερτελής in Nicomachus, ὑπερτέλειος in Theon), the sum of whose parts, i.e. submultiples, is greater than the number itself, e.g. 12, 24 etc., the sum of the parts of 12 being $6+4+3+2+1=16$, and the sum of the parts of 24 being $12+8+6+4+3+2+1=36$, (2) the *defective* (ἐλλιπής), the sum of whose parts is less than the whole, e.g. 8 or 14, the parts in the first case adding up to $4+2+1$, or 7, and in the second case to $7+2+1$, or 10. All three classes are however made by Theon subdivisions of numbers in general, but by Nicomachus subdivisions of *even* numbers.

The term *perfect* was used by the Pythagoreans, but in another sense, of 10; while Theon tells us (p. 46, 14) that 3 was also called perfect "because it is the first number that has beginning, middle and extremity; it is also both a *line* and a *plane* (for it is an equilateral triangle having each side made up of two units), and it is the first link and potentiality of the solid (for a solid must be conceived of in three dimensions)."

There are certain unexpressed axioms used in Book VII. as there are in earlier Books.

The following may be noted.

1. If A measures B, and B measures C, A will measure C.

2. If A measures B, and also measures C, A will measure the difference between B and C when they are unequal.

3. If A measures B, and also measures C, A will measure the sum of B and C.

It is clear, from what we know of the Pythagorean theory of numbers, of musical intervals expressed by numbers, of different kinds of *means* etc., that the substance of Euclid Books VII.—IX. was no new thing but goes back, at least, to the Pythagoreans. It is well known that the mathematics of Plato's *Timaeus* is essentially Pythagorean. It is therefore *a priori* probable (if not perhaps quite certain) that Plato πυθαγορίζει even in the passage (32 A, B) where he speaks of numbers "whether solid or square" in continued proportion, and proceeds to say that between *planes* one mean suffices, but to connect two *solids* two means are necessary. This passage has been much discussed, but I think that by "planes" and "solids" Plato certainly meant *square* and *solid numbers* respectively, so that the allusion must be to the theorems established in Eucl. VIII. 11, 12, that between two square numbers there is one mean proportional number, and between two cube numbers there are two mean proportional numbers[1].

[1] It is true that *similar* plane and solid numbers have the same property (Eucl. VIII. 18, 19); but, if Plato had meant similar plane and solid numbers generally, I think it would have been necessary to specify that they were "similar," whereas, seeing that the *Timaeus* is as a whole concerned with regular figures, there is nothing unnatural in allowing *regular* or *equilateral* to be understood. Further Plato speaks first of δυνάμεις and ὄγκοι and then of "planes" (ἐπίπεδα) and "solids" (στερεά) in such a way as to suggest that δυνάμεις correspond to ἐπίπεδα and ὄγκοι to στερεά. Now the regular meaning of δύναμις is *square* (or sometimes *square root*), and I think it is here used in the sense of *square*, notwithstanding that Plato seems to speak of *three* squares in continued proportion, whereas, in general, the mean between two squares as extremes would not be square but oblong. And, if δυνάμεις are squares, it is reasonable to suppose that the ὄγκοι are also *equilateral*, i.e. the "solids" are cubes. I am aware that Th. Häbler (*Bibliotheca Mathematica*, VIII$_3$, 1908, pp. 173—4) thinks that the passage is to be explained by reference to the problem of the duplication of the cube, and does not refer to numbers at all. Against this we have to put the evidence of Nicomachus (II. 24, 6) who, in speaking of "a certain Platonic theorem," quotes the very same results of Eucl. VIII. 11, 12. Secondly, it is worth noting that Häbler's explanation is distinctly ruled out by Democritus the Platonist (3rd cent. A.D.) who, according to Proclus

It is no less clear that, in his method and line of argument, Euclid was following earlier models, though no doubt making improvements in the exposition. The tract on the *Sectio Canonis*, κατατομη κανόνος (as to the genuineness of which see above, Vol. I., p. 17) is in style and in the form of the propositions generally akin to the *Elements*. In one proposition (2) the author says "*we learned* (ἐμάθομεν) that, if as many numbers as we please be in (continued) proportion, and the first measures the last, the first will also measure the intermediate numbers"; here he practically quotes *Elem.* VIII. 7. In the 3rd proposition he proves that no number can be a mean between two numbers in the ratio known as ἐπιμόριος, the ratio, that is, of $n+1$ to n, where n is any integer greater than unity. Now, fortunately, Boethius, *De institutione musica*, III. 11 (pp. 285—6, ed. Friedlein), has preserved a proof by Archytas of this same proposition; and the proof is substantially identical with that of Euclid. The two proofs are placed side by side in an article by Tannery (*Bibliotheca Mathematica*, VI_3, 1905/6, p. 227). Archytas writes the smaller term of the proportion first (instead of the greater, as Euclid does). Let, he says, A, B be the "superparticularis proportio" (ἐπιμόριον διάστημα in Euclid). Take C, DE the smallest numbers which are in the ratio of A to B. [Here DE means $D+E$: and in this respect the notation is different from that of Euclid who, as usual, takes a line DF divided into two parts at G, GF corresponding to E, and DG to D, in Archytas' notation. The step of taking C, DE, the smallest numbers in the ratio of A to B, presupposes Eucl. VII. 33.] Then DE exceeds C by an aliquot part of itself and of C [cf. the definition of ἐπιμόριος ἀριθμός in Nicomachus, I. 19, 1]. Let D be the excess [i.e. E is supposed equal to C]. "I say that D is not a number but an unit."

For, if D is a number and a part of DE, it measures DE; hence it measures E, that is, C. Thus D measures both C and DE, which is impossible; for the smallest numbers which are in the same ratio as any numbers are prime to one another. [This presupposes Eucl. VII. 22.] Therefore D is an unit; that is, DE exceeds C by an unit. Hence no number can be found which is a mean between two numbers C, DE. Therefore neither can any number be a mean between the original numbers A, B which are in the same ratio [this implies Eucl. VII. 20].

We have then here a clear indication of the existence at least as early as the date of Archytas (about 430—365 B.C.) of an *Elements of Arithmetic* in the form which we call Euclidean; and no doubt text-books of the sort existed even before Archytas, which probably Archytas himself and Eudoxus improved and developed in their turn.

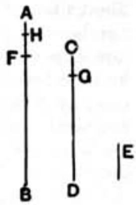
(*In Platonis Timaeum commentaria*, 149 C), said that the difficulties of the passage of the *Timaeus* had *misled* some people into connecting it with the duplication of the cube, whereas it really referred to *similar* planes and solids with sides in *rational numbers*. Thirdly, I do not think that, under the supposition that the Delian problem is referred to, we get the required sense. The problem in that case is not that of finding two mean proportionals *between two cubes* but that of finding a second cube the content of which shall be equal to twice, or k times (where k is any number not a complete cube), the content of a given cube (a^3). Two mean proportionals are found, not between cubes, but between two *straight lines* in the ratio of 1 to k, or between a and ka. Unless k is a cube, there would be no point in saying that two means are necessary to connect 1 and k, and not one mean; for $\sqrt[3]{k}$ is no more natural than $\sqrt{k}$, and would be less natural in the case where k happened to be square. On the other hand, if k is a cube, so that it is a question of finding means between *cube numbers*, the dictum of Plato is perfectly intelligible; nor is any real difficulty caused by the generality of the statement that two means are *always* necessary to connect them, because any property enunciated generally of two cube numbers should obviously be true of cubes *as such*, that is, it must hold in the extreme case of two cubes which are *prime to one another*.

BOOK VII. PROPOSITIONS.

PROPOSITION 1.

Two unequal numbers being set out, and the less being continually subtracted in turn from the greater, if the number which is left never measures the one before it until an unit is left, the original numbers will be prime to one another.

For, the less of two unequal numbers AB, CD being continually subtracted from the greater, let the number which is left never measure the one before it until an unit is left;



I say that AB, CD are prime to one another, that is, that an unit alone measures AB, CD.

For, if AB, CD are not prime to one another, some number will measure them.

Let a number measure them, and let it be E; let CD, measuring BF, leave FA less than itself,

let AF, measuring DG, leave GC less than itself,

and let GC, measuring FH, leave an unit HA.

Since, then, E measures CD, and CD measures BF,

therefore E also measures BF.

But it also measures the whole BA;

therefore it will also measure the remainder AF.

But AF measures DG;

therefore E also measures DG.

But it also measures the whole DC·

therefore it will also measure the remainder CG.

But CG measures FH;

therefore E also measures FH.

But it also measures the whole FA;

therefore it will also measure the remainder, the unit AH, though it is a number: which is impossible.

Therefore no number will measure the numbers AB, CD; therefore AB, CD are prime to one another. [VII. Def. 12]

Q. E. D.

It is proper to remark here that the representation in Books VII. to IX. of numbers by straight lines is adopted by Heiberg from the MSS. The method of those editors who substitute *points* for lines is open to objection because it practically necessitates, in many cases, the use of specific numbers, which is contrary to Euclid's manner.

"Let CD, measuring BF, leave FA less than itself." This is a neat abbreviation for saying, measure along BA successive lengths equal to CD until a point F is reached such that the length FA remaining is less than CD; in other words, let BF be the largest exact multiple of CD contained in BA.

Euclid's method in this proposition is an application to the particular case of prime numbers of the method of finding the greatest common measure of two numbers not prime to one another, which we shall find in the next proposition. With our notation, the method may be shown thus. Supposing the two numbers to be a, b, we have, say,

$$\begin{array}{l} b\,)\,a\,(\,p \\ \quad\underline{pb} \\ \quad c\,)\,b\,(\,q \\ \qquad\underline{qc} \\ \qquad d\,)\,c\,(\,r \\ \qquad\quad\underline{rd} \\ \qquad\quad 1 \end{array}$$

If now a, b are not prime to one another, they must have a common measure e, where e is some integer, not unity.

And since e measures a, b, it measures $a - pb$, i.e. c.

Again, since e measures b, c, it measures $b - qc$, i.e. d,

and lastly, since e measures c, d, it measures $c - rd$, i.e. 1:

which is impossible.

Therefore there is no integer, except unity, that measures a, b, which are accordingly prime to one another.

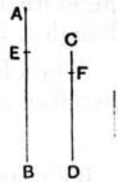
Observe that Euclid assumes as an axiom that, if a, b are both divisible by c, so is $a - pb$. In the next proposition he assumes as an axiom that c will in the case supposed divide $a + pb$.

PROPOSITION 2.

Given two numbers not prime to one another, to find their greatest common measure.

Let AB, CD be the two given numbers not prime to one another.

Thus it is required to find the greatest common measure of AB, CD.



If now CD measures AB—and it also measures itself—CD is a common measure of CD, AB.

And it is manifest that it is also the greatest; for no greater number than CD will measure CD.

But, if CD does not measure AB, then, the less of the numbers AB, CD being continually subtracted from the greater, some number will be left which will measure the one before it.

For an unit will not be left; otherwise AB, CD will be prime to one another [VII. 1], which is contrary to the hypothesis.

Therefore some number will be left which will measure the one before it.

Now let CD, measuring BE, leave EA less than itself, let EA, measuring DF, leave FC less than itself, and let CF measure AE.

Since then, CF measures AE, and AE measures DF, therefore CF will also measure DF.

But it also measures itself;

therefore it will also measure the whole CD.

But CD measures BE;

therefore CF also measures BE.

But it also measures EA;

therefore it will also measure the whole BA.

But it also measures CD;

therefore CF measures AB, CD.

Therefore CF is a common measure of AB, CD.

I say next that it is also the greatest.

For, if CF is not the greatest common measure of AB, CD, some number which is greater than CF will measure the numbers AB, CD.

Let such a number measure them, and let it be G.

Now, since G measures CD, while CD measures BE, G also measures BE.

But it also measures the whole BA;

therefore it will also measure the remainder AE.

But AE measures DF;

therefore G will also measure DF.

But it also measures the whole DC;

therefore it will also measure the remainder CF, that is, the greater will measure the less: which is impossible.

Therefore no number which is greater than CF will measure the numbers AB, CD;

therefore CF is the greatest common measure of AB, CD.

PORISM. From this it is manifest that, if a number measure two numbers, it will also measure their greatest common measure. Q. E. D.

Here we have the exact method of finding the greatest common measure given in the text-books of algebra, including the *reductio ad absurdum* proof that the number arrived at is not only a common measure but the *greatest* common measure. The process of finding the greatest common measure is simply shown thus:

$$\begin{array}{l} b\,)\,a\,(\,p \\ \quad\underline{pb} \\ \quad c\,)\,b\,(\,q \\ \qquad\underline{qc} \\ \qquad d\,)\,c\,(\,r \\ \qquad\quad\underline{rd} \end{array}$$

We shall arrive, says Euclid, at some number, say d, which measures the one before it, i.e. such that $c = rd$. Otherwise the process would go on until we arrived at unity. This is impossible because in that case a, b would be prime to one another, which is contrary to the hypothesis.

Next, like the text-books of algebra, he goes on to show that d will be *some* common measure of a, b. For d measures c;
therefore it measures $qc + d$, that is, b,
and hence it measures $pb + c$, that is, a.

Lastly, he proves that d is the *greatest* common measure of a, b as follows. Suppose that e is a common measure greater than d.

Then e, measuring a, b, must measure $a - pb$, or c.

Similarly e must measure $b-qc$, that is, d: which is impossible, since e is by hypothesis greater than d.

Therefore etc.

Euclid's proposition is thus *identical* with the algebraical proposition as generally given, e.g. in Todhunter's algebra, except that of course Euclid's numbers are integers.

Nicomachus gives the same rule (though without proving it) when he shows how to determine whether two given *odd* numbers are prime or not prime to one another, and, if they are not prime to one another, what is their common measure. We are, he says, to compare the numbers in turn by continually taking the less from the greater as many times as possible, then taking the remainder as many times as possible from the less of the original numbers, and so on; this process "will finish either at an unit or at some one and the same number," by which it is implied that the division of a greater number by a less is done by *separate subtractions* of the less. Thus, with regard to 21 and 49, Nicomachus says, "I subtract the less from the greater; 28 is left; then again I subtract from this the same 21 (for this is possible); 7 is left; I subtract this from 21, 14 is left; from which I again subtract 7 (for this is possible); 7 will be left, but 7 cannot be subtracted from 7." The last phrase is curious, but the meaning of it is obvious enough, as also the meaning of the phrase about ending "at one and the same number."

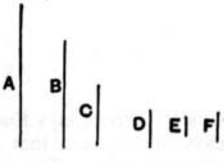
The proof of the Porism is of course contained in that part of the proposition which proves that G, a common measure different from CF, must measure CF. The supposition, thereby proved to be false, that G is greater than CF does not affect the validity of the proof that G measures CF in any case.

PROPOSITION 3.

Given three numbers not prime to one another, to find their greatest common measure.

Let A, B, C be the three given numbers not prime to one another;

thus it is required to find the greatest common measure of A, B, C.



For let the greatest common measure, D, of the two numbers A, B be taken; [VII. 2]

then D either measures, or does not measure, C.

First, let it measure it.

But it measures A, B also;

therefore D measures A, B, C;

therefore D is a common measure of A, B, C.

I say that it is also the greatest.

For, if D is not the greatest common measure of A, B, C, some number which is greater than D will measure the numbers A, B, C.

Let such a number measure them, and let it be E.

Since then E measures A, B, C,
it will also measure A, B;
therefore it will also measure the greatest common measure of A, B. [VII. 2, Por.]

But the greatest common measure of A, B is D;
therefore E measures D, the greater the less: which is impossible.

Therefore no number which is greater than D will measure the numbers A, B, C;

therefore D is the greatest common measure of A, B, C.

Next, let D not measure C;

I say first that C, D are not prime to one another.

For, since A, B, C are not prime to one another, some number will measure them.

Now that which measures A, B, C will also measure A, B, and will measure D, the greatest common measure of A, B. [VII. 2, Por.]

But it measures C also;
therefore some number will measure the numbers D, C;
therefore D, C are not prime to one another.

Let then their greatest common measure E be taken. [VII. 2]

Then, since E measures D,
and D measures A, B,
therefore E also measures A, B.

But it measures C also;
therefore E measures A, B, C;
therefore E is a common measure of A, B, C.

I say next that it is also the greatest.

For, if E is not the greatest common measure of A, B, C, some number which is greater than E will measure the numbers A, B, C.

Let such a number measure them, and let it be F.

Now, since F measures A, B, C,
it also measures A, B;
therefore it will also measure the greatest common measure of A, B. [VII. 2, Por.]

But the greatest common measure of A, B is D;
therefore F measures D.

And it measures C also;
therefore F measures D, C;
therefore it will also measure the greatest common measure of D, C. [VII. 2, Por.]

But the greatest common measure of D, C is E;
therefore F measures E, the greater the less: which is impossible.

Therefore no number which is greater than E will measure the numbers A, B, C;
therefore E is the greatest common measure of A, B, C.

Q. E. D.

Euclid's proof is here longer than we should make it because he distinguishes two cases, the simpler of which is really included in the other.

Having taken the greatest common measure, say d, of a, b, two of the three given numbers a, b, c, he distinguishes the cases

(1) in which d measures c,

(2) in which d does not measure c.

In the first case the greatest common measure of d, c is d itself; in the second case it has to be found by a repetition of the process of VII. 2. In either case the greatest common measure of a, b, c is the greatest common measure of d, c.

But, after disposing of the simpler case, Euclid thinks it necessary to prove that, if d does not measure c, d and c must necessarily *have* a greatest common measure. This he does by means of the original hypothesis that a, b, c are not prime to one another. Since they are not prime to one another, they must have a common measure; any common measure of a, b is a measure of d, and therefore any common measure of a, b, c is a common measure of d, c; hence d, c must have a common measure, and are therefore not prime to one another.

The proofs of cases (1) and (2) repeat exactly the same argument as we saw in VII. 2, and it is proved separately for d in case (1) and e in case (2), where e is the greatest common measure of d, c,

(α) that it is a common measure of a, b, c,

(β) that it is the *greatest* common measure.

Heron remarks (an-Nairīzī, ed. Curtze, p. 191) that the method does not only enable us to find the greatest common measure of *three* numbers; it can be used to find the greatest common measure of as many numbers as we please. This is because any number measuring two numbers also measures their greatest common measure; and hence we can find the G.C.M. of pairs, then the G.C.M. of pairs of these, and so on, until only two numbers are left and we find the G.C.M. of these. Euclid tacitly assumes this extension in VII. 33, where he takes the greatest common measure of *as many numbers as we please.*

PROPOSITION 4.

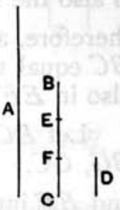
Any number is either a part or parts of any number, the less of the greater.

Let A, BC be two numbers, and let BC be the less; I say that BC is either a part, or parts, of A.

For A, BC are either prime to one another or not.

First, let A, BC be prime to one another.

Then, if BC be divided into the units in it, each unit of those in BC will be some part of A; so that BC is parts of A.



Next let A, BC not be prime to one another; then BC either measures, or does not measure, A.

If now BC measures A, BC is a part of A.

But, if not, let the greatest common measure D of A, BC be taken; [VII. 2]

and let BC be divided into the numbers equal to D, namely BE, EF, FC.

Now, since D measures A, D is a part of A.

But D is equal to each of the numbers BE, EF, FC; therefore each of the numbers BE, EF, FC is also a part of A; so that BC is parts of A.

Therefore etc.

Q. E. D.

The meaning of the enunciation is of course that, if a, b be two numbers of which b is the less, then b is either a *submultiple* or *some proper fraction* of a.

(1) If a, b are prime to one another, divide each into its units; then b contains b of the same parts of which a contains a. Therefore b is "parts" or a *proper fraction* of a.

(2) If a, b be not prime to one another, either b measures a, in which case b is a submultiple or "part" of a, or, if g be the greatest common measure of a, b, we may put $a = mg$ and $b = ng$, and b will contain n of the same parts (g) of which a contains m, so that b is again "parts," or a *proper fraction*, of a.

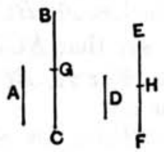
Proposition 5.

If a number be a part of a number, and another be the same part of another, the sum will also be the same part of the sum that the one is of the one.

For let the number A be a part of BC,

and another, D, the same part of another EF that A is of BC;

I say that the sum of A, D is also the same part of the sum of BC, EF that A is of BC.



For since, whatever part A is of BC, D is also the same part of EF,

therefore, as many numbers as there are in BC equal to A, so many numbers are there also in EF equal to D.

Let BC be divided into the numbers equal to A, namely BG, GC,

and EF into the numbers equal to D, namely EH, HF;

then the multitude of BG, GC will be equal to the multitude of EH, HF.

And, since BG is equal to A, and EH to D,

therefore BG, EH are also equal to A, D.

For the same reason

GC, HF are also equal to A, D.

Therefore, as many numbers as there are in BC equal to A, so many are there also in BC, EF equal to A, D.

Therefore, whatever multiple BC is of A, the same multiple also is the sum of BC, EF of the sum of A, D.

Therefore, whatever part A is of BC, the same part also is the sum of A, D of the sum of BC, EF.

Q. E. D.

If $\quad a = \frac{1}{n} b$, and $c = \frac{1}{n} d$, then

$$a + c = \frac{1}{n}(b + d).$$

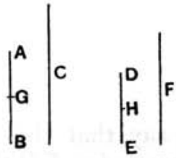
The proposition is of course true for any quantity of pairs of numbers similarly related, as is the next proposition also; and both propositions are used in the extended form in VII. 9, 10.

Proposition 6.

If a number be parts of a number, and another be the same parts of another, the sum will also be the same parts of the sum that the one is of the one.

For let the number AB be parts of the number C, and another, DE, the same parts of another, F, that AB is of C;

I say that the sum of AB, DE is also the same parts of the sum of C, F that AB is of C.



For since, whatever parts AB is of C, DE is also the same parts of F,

therefore, as many parts of C as there are in AB, so many parts of F are there also in DE.

Let AB be divided into the parts of C, namely AG, GB, and DE into the parts of F, namely DH, HE;

thus the multitude of AG, GB will be equal to the multitude of DH, HE.

And since, whatever part AG is of C, the same part is DH of F also,

therefore, whatever part AG is of C, the same part also is the sum of AG, DH of the sum of C, F. [VII. 5]

For the same reason,

whatever part GB is of C, the same part also is the sum of GB, HE of the sum of C, F.

Therefore, whatever parts AB is of C, the same parts also is the sum of AB, DE of the sum of C, F.

Q. E. D.

If $$a = \frac{m}{n} b, \text{ and } c = \frac{m}{n} d,$$

then $$a + c = \frac{m}{n}(b + d).$$

More generally, if

$$a = \frac{m}{n} b,\ c = \frac{m}{n} d,\ e = \frac{m}{n} f, \ldots$$

then $$(a + c + e + g + \ldots) = \frac{m}{n}(b + d + f + h + \ldots).$$

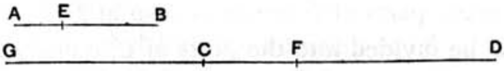
In Euclid's proposition $m < n$, but the generality of the result is of course not affected. This proposition and the last are complementary to v. 1, which proves the corresponding result with *multiple* substituted for "*part*" or "*parts.*"

PROPOSITION 7.

If a number be that part of a number, which a number subtracted is of a number subtracted, the remainder will also be the same part of the remainder that the whole is of the whole.

For let the number AB be that part of the number CD which AE subtracted is of CF subtracted;

I say that the remainder EB is also the same part of the remainder FD that the whole AB is of the whole CD.



For, whatever part AE is of CF, the same part also let EB be of CG.

Now since, whatever part AE is of CF, the same part also is EB of CG,

therefore, whatever part AE is of CF, the same part also is AB of GF. [VII. 5]

But, whatever part AE is of CF, the same part also, by hypothesis, is AB of CD;

therefore, whatever part AB is of GF, the same part is it of CD also;

therefore GF is equal to CD.

Let CF be subtracted from each;

therefore the remainder GC is equal to the remainder FD.

Now since, whatever part AE is of CF, the same part also is EB of GC,

while GC is equal to FD,

therefore, whatever part AE is of CF, the same part also is EB of FD.

But, whatever part AE is of CF, the same part also is AB of CD;

therefore also the remainder EB is the same part of the remainder FD that the whole AB is of the whole CD.

Q. E. D.

If $a = \frac{1}{n} b$ and $c = \frac{1}{n} d$, we are to prove that

$$a - c = \frac{1}{n}(b - d),$$

a result differing from that of VII. 5 in that *minus* is substituted for *plus*. Euclid's method is as follows.

Suppose that e is taken such that

$$a - c = \frac{1}{n} e. \quad \text{.................................}(1)$$

Now $$c = \frac{1}{n} d.$$

Therefore $$a = \frac{1}{n}(d + e), \qquad [\text{VII. } 5]$$

whence, from the hypothesis, $d + e = b$,

so that $e = b - d$,

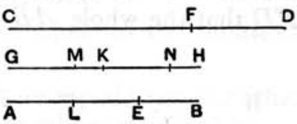
and, substituting this value of e in (1), we have

$$a - c = \frac{1}{n}(b - d).$$

Proposition 8.

If a number be the same parts of a number that a number subtracted is of a number subtracted, the remainder will also be the same parts of the remainder that the whole is of the whole.

For let the number AB be the same parts of the number CD that AE subtracted is of CF subtracted;



I say that the remainder EB is also the same parts of the remainder FD that the whole AB is of the whole CD.

For let GH be made equal to AB.

Therefore, whatever parts GH is of CD, the same parts also is AE of CF.

Let GH be divided into the parts of CD, namely GK, KH, and AE into the parts of CF, namely AL, LE;
thus the multitude of GK, KH will be equal to the multitude of AL, LE.

Now since, whatever part GK is of CD, the same part also is AL of CF,
while CD is greater than CF,
therefore GK is also greater than AL.

Let GM be made equal to AL.

Therefore, whatever part GK is of CD, the same part also is GM of CF;
therefore also the remainder MK is the same part of the remainder FD that the whole GK is of the whole CD. [VII. 7]

Again, since, whatever part KH is of CD, the same part also is EL of CF,
while CD is greater than CF,
therefore HK is also greater than EL.

Let KN be made equal to EL.

Therefore, whatever part KH is of CD, the same part also is KN of CF;
therefore also the remainder NH is the same part of the remainder FD that the whole KH is of the whole CD. [VII. 7]

But the remainder MK was also proved to be the same part of the remainder FD that the whole GK is of the whole CD;
therefore also the sum of MK, NH is the same parts of DF that the whole HG is of the whole CD.

But the sum of MK, NH is equal to EB,
and HG is equal to BA;
therefore the remainder EB is the same parts of the remainder FD that the whole AB is of the whole CD.

Q. E. D.

If $$a = \frac{m}{n} b \text{ and } c = \frac{m}{n} d, \qquad (m < n)$$

then $$a - c = \frac{m}{n}(b - d).$$

Euclid's proof amounts to the following.

Take e equal to $\frac{1}{n} b$, and f equal to $\frac{1}{n} d$.

Then since, by hypothesis, $b > d$,

$$e > f,$$

and, by VII. 7, $$e - f = \frac{1}{n}(b - d).$$

Repeat this for all the parts equal to e and f that there are in a, b respectively, and we have, by addition (a, b containing m of such parts respectively),

$$m(e-f)=\frac{m}{n}(b-d).$$

But $$m(e-f)=a-c.$$

Therefore $$a-c=\frac{m}{n}(b-d).$$

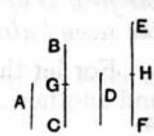
The propositions VII. 7, 8 are complementary to V. 5 which gives the corresponding result with *multiple* in the place of "part" or "parts."

Proposition 9.

If a number be a part of a number, and another be the same part of another, alternately also, whatever part or parts the first is of the third, the same part, or the same parts, will the second also be of the fourth.

For let the number A be a part of the number BC, and another, D, the same part of another, EF, that A is of BC;

I say that, alternately also, whatever part or parts A is of D, the same part or parts is BC of EF also.



For since, whatever part A is of BC, the same part also is D of EF,

therefore, as many numbers as there are in BC equal to A, so many also are there in EF equal to D.

Let BC be divided into the numbers equal to A, namely BG, GC,

and EF into those equal to D, namely EH, HF;

thus the multitude of BG, GC will be equal to the multitude of EH, HF.

Now, since the numbers BG, GC are equal to one another, and the numbers EH, HF are also equal to one another,

while the multitude of BG, GC is equal to the multitude of EH, HF,

therefore, whatever part or parts BG is of EH, the same part or the same parts is GC of HF also;

so that, in addition, whatever part or parts BG is of EH, the same part also, or the same parts, is the sum BC of the sum EF. [VII. 5, 6]

But BG is equal to A, and EH to D;
therefore, whatever part or parts A is of D, the same part or the same parts is BC of EF also.

Q. E. D.

If $a = \frac{1}{n} b$ and $c = \frac{1}{n} d$, then, whatever fraction ("part" or "parts") a is of c, the same fraction will b be of d.

Dividing b into each of its parts equal to a, and d into each of its parts equal to c, it is clear that, whatever fraction one of the parts a is of one of the parts c, the same fraction is any other of the parts a of any other of the parts c.

And the number of the parts a is equal to the number of the parts c, viz. n.

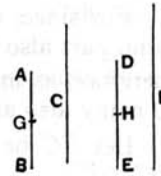
Therefore, by VII. 5, 6, na is the same fraction of nc that a is of c, i.e. b is the same fraction of d that a is of c.

Proposition 10.

If a number be parts of a number, and another be the same parts of another, alternately also, whatever parts or part the first is of the third, the same parts or the same part will the second also be of the fourth.

For let the number AB be parts of the number C, and another, DE, the same parts of another, F;

I say that, alternately also, whatever parts or part AB is of DE, the same parts or the same part is C of F also.



For since, whatever parts AB is of C, the same parts also is DE of F,

therefore, as many parts of C as there are in AB, so many parts also of F are there in DE.

Let AB be divided into the parts of C, namely AG, GB, and DE into the parts of F, namely DH, HE;

thus the multitude of AG, GB will be equal to the multitude of DH, HE.

Now since, whatever part AG is of C, the same part also is DH of F,

alternately also, whatever part or parts AG is of DH, the same part or the same parts is C of F also. [VII. 9]

For the same reason also,

whatever part or parts GB is of HE, the same part or the same parts is C of F also;

so that, in addition, whatever parts or part AB is of DE, the same parts also, or the same part, is C of F. [VII. 5, 6]

Q. E. D.

If $a = \frac{m}{n} b$ and $c = \frac{m}{n} d$, then, whatever fraction a is of c, the same fraction is b of d.

To prove this, a is divided into its m parts equal to b/n, and c into its m parts equal to d/n.

Then, by VII. 9, whatever fraction one of the m parts of a is of one of the m parts of c, the same fraction is b of d.

And, by VII. 5, 6, whatever fraction one of the m parts of a is of one of the m parts of c, the same fraction is the sum of the parts of a (that is, a) of the sum of the parts of c (that is, c).

Whence the result follows.

In the Greek text, after the words "so that, in addition" in the last line but one, is an additional explanation making the reference to VII. 5, 6 clearer, as follows: "whatever part or parts AG is of DH, the same part or the same parts is GB of HE also;

therefore also, whatever part or parts AG is of DH, the same part or the same parts is AB of DE also. [VII. 5, 6]

But it was proved that, whatever part or parts AG is of DH, the same part or the same parts is C of F also;

therefore also" etc. as in the last two lines of the text.

Heiberg concludes, on the authority of P, which only has the words in the margin in a later hand, that they may be attributed to Theon.

Proposition 11.

If, as whole is to whole, so is a number subtracted to a number subtracted, the remainder will also be to the remainder as whole to whole.

As the whole AB is to the whole CD, so let AE subtracted be to CF subtracted;

I say that the remainder EB is also to the remainder FD as the whole AB to the whole CD.

Since, as AB is to CD, so is AE to CF, whatever part or parts AB is of CD, the same part or the same parts is AE of CF also; [VII. Def. 20]

Therefore also the remainder EB is the same part or parts of FD that AB is of CD. [VII. 7, 8]

Therefore, as EB is to FD, so is AB to CD. [VII. Def. 20]

Q. E. D.

It will be observed that, in dealing with the proportions in Props. 11—13, Euclid only contemplates the case where the first number is "a part" or "parts" of the second, while in Prop. 13 he assumes the first to be "a part" or "parts" of the third also; that is, the first number is in all three propositions assumed to be less than the second, and in Prop. 13 less than the third also. Yet the figures in Props. 11 and 13 are inconsistent with these assumptions. If the facts are taken to correspond to the figures in these propositions, it is necessary to take account of the other possibilities involved in the definition of proportion (VII. Def. 20), that the first number may also be a multiple, or a multiple *plus* "a part" or "parts" (including *once* as a multiple in this case), of each number with which it is compared. Thus a number of different cases would have to be considered. The remedy is to make the ratio which is in the lower terms the first ratio, and to invert the ratios, if necessary, in order to make "a part" or "parts" literally apply.

If $a : b = c : d, \quad (a > c,\ b > d)$

then $(a - c) : (b - d) = a : b.$

This proposition for numbers corresponds to V. 19 for magnitudes. The enunciation is the same except that the masculine (agreeing with ἀριθμός) takes the place of the neuter (agreeing with μέγεθος).

The proof is no more than a combination of the arithmetical definition of proportion (VII. Def. 20) with the results of VII. 7, 8. The language of proportions is turned into the language of fractions by Def. 20; the results of VII. 7, 8 are then used and the language retransformed by Def. 20 into the language of proportions.

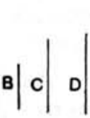
PROPOSITION 12.

If there be as many numbers as we please in proportion, then, as one of the antecedents is to one of the consequents, so are all the antecedents to all the consequents.

Let A, B, C, D be as many numbers as we please in proportion, so that,

as A is to B, so is C to D;

I say that, as A is to B, so are A, C to B, D.



For since, as A is to B, so is C to D, whatever part or parts A is of B, the same part or parts is C of D also. [VII. Def. 20]

Therefore also the sum of A, C is the same part or the same parts of the sum of B, D that A is of B. [VII. 5, 6]

Therefore, as A is to B, so are A, C to B, D. [VII. Def. 20]

If $a : a' = b : b' = c : c' = \dots,$

then each ratio is equal to $(a + b + c + \dots) : (a' + b' + c' + \dots)$.

The proposition corresponds to V. 12, and the enunciation is word for word the same with that of V. 12 except that ἀριθμός takes the place of μέγεθος.

Again the proof merely connects the arithmetical definition of proportion (VII. Def. 20) with the results of VII. 5, 6, which are quoted as true for any number of numbers, and not merely for two numbers as in the enunciations of VII. 5, 6.

Proposition 13.

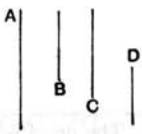
If four numbers be proportional, they will also be proportional alternately.

Let the four numbers A, B, C, D be proportional, so that,

as A is to B, so is C to D;

I say that they will also be proportional alternately, so that,

as A is to C, so will B be to D.



For since, as A is to B, so is C to D, therefore, whatever part or parts A is of B, the same part or the same parts is C of D also. [VII. Def. 20]

Therefore, alternately, whatever part or parts A is of C, the same part or the same parts is B of D also. [VII. 10]

Therefore, as A is to C, so is B to D. [VII. Def. 20]

Q. E. D.

If $\quad a : b = c : d$,

then, alternately, $\quad a : c = b : d$.

The proposition corresponds to V. 16 for magnitudes, and the proof consists in connecting VII. Def. 20 with the result of VII. 10.

Proposition 14.

If there be as many numbers as we please, and others equal to them in multitude, which taken two and two are in the same ratio, they will also be in the same ratio **ex aequali.**

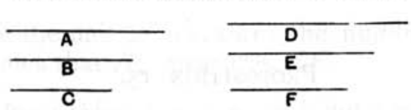
Let there be as many numbers as we please A, B, C, and others equal to them in multitude D, E, F, which taken two and two are in the same ratio, so that,

as A is to B, so is D to E,

and, as B is to C, so is E to F;

I say that, *ex aequali*,

as A is to C, so also is D to F.



For, since, as A is to B, so is D to E, therefore, alternately,

as A is to D, so is B to E. [VII. 13]

Again, since, as B is to C, so is E to F,
therefore, alternately,

as B is to E, so is C to F. [VII. 13]

But, as B is to E, so is A to D;
therefore also, as A is to D, so is C to F.

Therefore, alternately,

as A is to C, so is D to F. [*id.*]

If $a : b = d : e$,
and $b : c = e : f$,
then, *ex aequali*, $a : c = d : f$;
and the same is true however many successive numbers are so related.

The proof is simplicity itself.

By VII. 13, alternately, $a : d = b : e$,
and $b : e = c : f$.
Therefore $a : d = c : f$,
and, again alternately, $a : c = d : f$.

Observe that this simple method cannot be used to prove the corresponding proposition for magnitudes, V. 22, although V. 22 has been preceded by the two propositions in that Book corresponding to the propositions used here, viz. V. 16 and V. 11. The reason of this is that this method would only prove V. 22 for six magnitudes *all of the same kind*, whereas the magnitudes in V. 22 are not subject to this limitation.

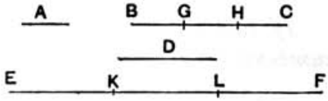
Heiberg remarks in a note on VII. 19 that, while Euclid has proved several propositions of Book V. over again, by a separate proof, for numbers, he has neglected to do so in certain cases; e.g., he often uses V. 11 in these propositions of Book VII., V. 9 in VII. 19, V. 7 in the same proposition, and so on. Thus Heiberg would apparently suppose Euclid to use V. 11 in the last step of the present proof (*Ratios which are the same with the same ratio are also the same with one another*). I think it preferable to suppose that Euclid regarded the last step as axiomatic; since, by the definition of proportion, the first number is the same multiple or the same part or the same parts of the second that the third is of the fourth: the assumption is no more than an assumption that the numbers or proper fractions which are respectively equal to the same number or proper fraction are equal to one another.

Though the proposition is only proved of six numbers, the extension to as many as we please (as expressed in the enunciation) is obvious.

PROPOSITION 15.

If an unit measure any number, and another number measure any other number the same number of times, alternately also, the unit will measure the third number the same number of times that the second measures the fourth.

For let the unit A measure any number BC, and let another number D measure any other number EF the same number of times;



I say that, alternately also, the unit A measures the number D the same number of times that BC measures EF.

For, since the unit A measures the number BC the same number of times that D measures EF,

therefore, as many units as there are in BC, so many numbers equal to D are there in EF also.

Let BC be divided into the units in it, BG, GH, HC, and EF into the numbers EK, KL, LF equal to D.

Thus the multitude of BG, GH, HC will be equal to the multitude of EK, KL, LF.

And, since the units BG, GH, HC are equal to one another, and the numbers EK, KL, LF are also equal to one another, while the multitude of the units BG, GH, HC is equal to the multitude of the numbers EK, KL, LF,

therefore, as the unit BG is to the number EK, so will the unit GH be to the number KL, and the unit HC to the number LF.

Therefore also, as one of the antecedents is to one of the consequents, so will all the antecedents be to all the consequents; [VII. 12]

therefore, as the unit BG is to the number EK, so is BC to EF.

But the unit BG is equal to the unit A,

and the number EK to the number D.

Therefore, as the unit A is to the number D, so is BC to EF.

Therefore the unit A measures the number D the same number of times that BC measures EF. Q. E. D.

If there be four numbers 1, m, a, ma (such that 1 measures m the same number of times that a measures ma), 1 measures a the same number of times that m measures ma.

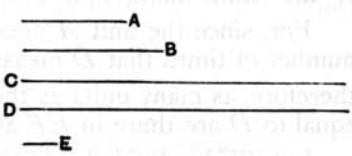
Except that the first number is unity and the numbers are said to *measure* instead of being a *part* of others, this proposition and its proof do not differ from VII. 9; in fact this proposition is a particular case of the other.

Proposition 16.

If two numbers by multiplying one another make certain numbers, the numbers so produced will be equal to one another.

Let A, B be two numbers, and let A by multiplying B make C, and B by multiplying A make D;

I say that C is equal to D.



For, since A by multiplying B has made C, therefore B measures C according to the units in A.

But the unit E also measures the number A according to the units in it;

therefore the unit E measures A the same number of times that B measures C.

Therefore, alternately, the unit E measures the number B the same number of times that A measures C. [VII. 15]

Again, since B by multiplying A has made D, therefore A measures D according to the units in B.

But the unit E also measures B according to the units in it;

therefore the unit E measures the number B the same number of times that A measures D.

But the unit E measured the number B the same number of times that A measures C;

therefore A measures each of the numbers C, D the same number of times.

Therefore C is equal to D. Q. E. D.

2. **The numbers so produced.** The Greek has οἱ γενόμενοι ἐξ αὐτῶν, "the (numbers) produced *from them*." By "from them" Euclid means "from the original numbers," though this is not very clear even in the Greek. I think ambiguity is best avoided by leaving out the words.

This proposition proves that, *if any numbers be multiplied together, the order of multiplication is indifferent*, or $ab = ba$.

It is important to get a clear understanding of what Euclid means when he speaks of *one number multiplying another*. VII. Def. 15 states that the effect of "a multiplying b" is taking a times b. We shall always represent "a times b" by ab and "b times a" by ba. This being premised, the proof that $ab = ba$ may be represented as follows in the language of proportions.

By VII. Def. 20,	$1 : a = b : ab$.	
Therefore, alternately,	$1 : b = a : ab$.	[VII. 13]
Again, by VII. Def. 20,	$1 : b = a : ba$.	
Therefore	$a : ab = a : ba$,	
or	$ab = ba$.	

Euclid does not use the language of proportions but that of fractions or their equivalent measures, quoting VII. 15, a particular case of VII. 13 differently expressed, instead of VII. 13 itself.

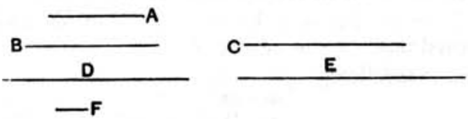
PROPOSITION 17.

If a number by multiplying two numbers make certain numbers, the numbers so produced will have the same ratio as the numbers multiplied.

For let the number A by multiplying the two numbers B, C make D, E;

I say that, as B is to C, so is D to E.

For, since A by multiplying B has made D, therefore B measures D according to the units in A.



But the unit F also measures the number A according to the units in it;

therefore the unit F measures the number A the same number of times that B measures D.

Therefore, as the unit F is to the number A, so is B to D. [VII. Def. 20]

For the same reason,

as the unit F is to the number A, so also is C to E;

therefore also, as B is to D, so is C to E.

Therefore, alternately, as B is to C, so is D to E. [VII. 13]

Q. E. D.

$$b : c = ab : ac.$$

In this case Euclid translates the language of measures into that of proportions, and the proof is exactly like that set out in the last note.

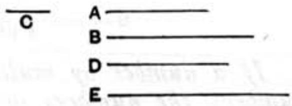
By VII. Def. 20,	$1 : a = b : ab$,	
and	$1 : a = c : ac$.	
Therefore	$b : ab = c : ac$,	
and, alternately,	$b : c = ab : ac$.	[VII. 13]

PROPOSITION 18.

If two numbers by multiplying any number make certain numbers, the numbers so produced will have the same ratio as the multipliers.

For let two numbers A, B by multiplying any number C make D, E;

I say that, as A is to B, so is D to E.



For, since A by multiplying C has made D,

therefore also C by multiplying A has made D. [VII. 16]

For the same reason also

C by multiplying B has made E.

Therefore the number C by multiplying the two numbers A, B has made D, E.

Therefore, as A is to B, so is D to E. [VII. 17]

It is here proved that $a : b = ac : bc$.

The argument is as follows.

$ac = ca$. [VII. 16]

Similarly $bc = cb$.

And $a : b = ca : cb$; [VII. 17]

therefore $a : b = ac : bc$.

PROPOSITION 19.

If four numbers be proportional, the number produced from the first and fourth will be equal to the number produced from the second and third; and, if the number produced from the first and fourth be equal to that produced from the second and third, the four numbers will be proportional.

Let A, B, C, D be four numbers in proportion, so that,

as A is to B, so is C to D;

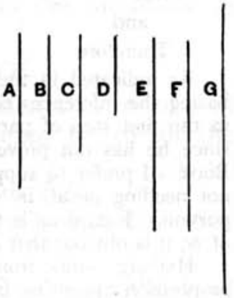
and let A by multiplying D make E, and let B by multiplying C make F;

I say that E is equal to F.

For let A by multiplying C make G.

Since, then, A by multiplying C has made G, and by multiplying D has made E,
the number A by multiplying the two numbers C, D has made G, E.

Therefore, as C is to D, so is G to E. [VII. 17]



But, as C is to D, so is A to B;
therefore also, as A is to B, so is G to E.

Again, since A by multiplying C has made G,
but, further, B has also by multiplying C made F,
the two numbers A, B by multiplying a certain number C have made G, F.

Therefore, as A is to B, so is G to F. [VII. 18]

But further, as A is to B, so is G to E also;
therefore also, as G is to E, so is G to F.

Therefore G has to each of the numbers E, F the same ratio;

therefore E is equal to F. [cf. V. 9]

Again, let E be equal to F;
I say that, as A is to B, so is C to D.

For, with the same construction,
since E is equal to F,
therefore, as G is to E, so is G to F. [cf. V. 7]

But, as G is to E, so is C to D, [VII. 17]
and, as G is to F, so is A to B. [VII. 18]

Therefore also, as A is to B, so is C to D.

Q. E. D.

If $a : b = c : d$,

then $ad = bc$; and conversely.

The proof is equivalent to the following.

(1) $ac : ad = c : d$ [VII. 17]

$= a : b$.

But $a : b = ac : bc$. [VII. 18]

Therefore $ac : ad = ac : bc$,

or $ad = bc$.

(2) Since $ad = bc$,

$$ac : ad = ac : bc.$$

But $ac : ad = c : d$, [VII. 17]

and $ac : bc = a : b$. [VII. 18]

Therefore $a : b = c : d$.

As indicated in the note on VII. 14 above, Heiberg regards Euclid as basing the inferences contained in the last step of part (1) of this proof and in the first step of part (2) on the propositions V. 9 and V. 7 respectively, since he has not proved those propositions separately for numbers in this Book. I prefer to suppose that he regarded the inferences as obvious and not needing proof, in view of the definition of numbers which are in proportion. E.g., if ac is the same fraction ("part" or "parts") of ad that ac is of bc, it is obvious that ad must be equal to bc.

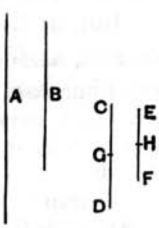
Heiberg omits from his text here, and relegates to an Appendix, a proposition appearing in the manuscripts V, p, ϕ to the effect that, if *three* numbers be proportional, the product of the extremes is equal to the square of the mean, and conversely. It does not appear in P in the first hand, B has it in the margin only, and Campanus omits it, remarking that Euclid does not give the proposition about *three* proportionals as he does in VI. 17, since it is easily proved by the proposition just given. Moreover an-Nairīzī quotes the proposition about three proportionals *as an observation on* VII. 19 probably due to Heron (who is mentioned by name in the preceding paragraph).

PROPOSITION 20.

The least numbers of those which have the same ratio with them measure those which have the same ratio the same number of times, the greater the greater and the less the less.

For let CD, EF be the least numbers of those which have the same ratio with A, B;

I say that CD measures A the same number of times that EF measures B.



Now CD is not parts of A.

For, if possible, let it be so;

therefore EF is also the same parts of B that CD is of A. [VII. 13 and Def. 20]

Therefore, as many parts of A as there are in CD, so many parts of B are there also in EF.

Let CD be divided into the parts of A, namely CG, GD, and EF into the parts of B, namely EH, HF;

thus the multitude of CG, GD will be equal to the multitude of EH, HF.

Now, since the numbers CG, GD are equal to one another, and the numbers EH, HF are also equal to one another, while the multitude of CG, GD is equal to the multitude of EH, HF,

therefore, as CG is to EH, so is GD to HF.

Therefore also, as one of the antecedents is to one of the consequents, so will all the antecedents be to all the consequents. [VII. 12]

Therefore, as CG is to EH, so is CD to EF.

Therefore CG, EH are in the same ratio with CD, EF, being less than they:

which is impossible, for by hypothesis CD, EF are the least numbers of those which have the same ratio with them.

Therefore CD is not parts of A;

therefore it is a part of it. [VII. 4]

And EF is the same part of B that CD is of A; [VII. 13 and Def. 20]

therefore CD measures A the same number of times that EF measures B.

Q. E. D.

If a, b are the least numbers among those which have the same ratio (i.e. if a/b is a fraction in its lowest terms), and c, d are any others in the same ratio, i.e. if

$$a : b = c : d,$$

then $a = \frac{1}{n} c$ and $b = \frac{1}{n} d$, where n is some integer.

The proof is by *reductio ad absurdum*, thus.

[Since $a < c$, a is some proper fraction ("part" or "parts") of c, by VII. 4.]

Now a cannot be equal to $\frac{m}{n} c$, where m is an integer less than n but greater than 1.

For, if $a = \frac{m}{n} c$, $b = \frac{m}{n} d$ also. [VII. 13 and Def. 20]

Take each of the m parts of a with each of the m parts of b, two and two; the ratio of the members of all pairs is the same ratio $\frac{1}{m} a : \frac{1}{m} b$.

Therefore

$$\frac{1}{m} a : \frac{1}{m} b = a : b. \qquad \text{[VII. 12]}$$

But $\frac{1}{m} a$ and $\frac{1}{m} b$ are respectively less than a, b and they are in the same ratio: which contradicts the hypothesis.

Hence a can only be "a part" of c, or

$$a \text{ is of the form } \frac{1}{n}c,$$

and therefore $$b \text{ is of the form } \frac{1}{n}d.$$

Here also Heiberg omits a proposition which was no doubt interpolated by Theon (B, V, p, ϕ have it as VII. 22, but P only has it in the margin and in a later hand; Campanus also omits it) proving for numbers the *ex aequali* proposition when "the proportion is perturbed," i.e. (cf. enunciation of V. 22) if

$$a : b = e : f, \quad \text{.........................(1)}$$

and $$b : c = d : e, \quad \text{.........................(2)}$$

then $$a : c = d : f.$$

The proof (see Heiberg's Appendix) depends on VII. 19.

From (1) we have $af = be$,

and from (2) $be = cd$. [VII. 19]

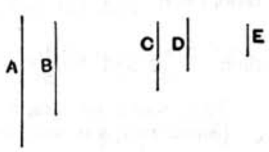
Therefore $af = cd$,

and accordingly $a : c = d : f$. [VII. 19]

Proposition 21.

Numbers prime to one another are the least of those which have the same ratio with them.

Let A, B be numbers prime to one another; I say that A, B are the least of those which have the same ratio with them.



For, if not, there will be some numbers less than A, B which are in the same ratio with A, B.

Let them be C, D.

Since, then, the least numbers of those which have the same ratio measure those which have the same ratio the same number of times, the greater the greater and the less the less, that is, the antecedent the antecedent and the consequent the consequent, [VII. 20]

therefore C measures A the same number of times that D measures B.

Now, as many times as C measures A, so many units let there be in E.

Therefore D also measures B according to the units in E.

And, since C measures A according to the units in E, therefore E also measures A according to the units in C. [VII. 16]

For the same reason

E also measures B according to the units in D. [VII. 16]

Therefore E measures A, B which are prime to one another: which is impossible. [VII. Def. 12]

Therefore there will be no numbers less than A, B which are in the same ratio with A, B.

Therefore A, B are the least of those which have the same ratio with them.

Q. E. D.

In other words, if a, b are prime to one another, the ratio $a : b$ is "in its lowest terms."

The proof is equivalent to the following.

If not, suppose that c, d are the *least* numbers for which

$$a : b = c : d.$$

[Euclid only supposes *some* numbers c, d in the ratio of a to b such that $c < a$, and (consequently) $d < b$. It is however necessary to suppose that c, d are the *least* numbers in that ratio in order to enable VII. 20 to be used in the proof.]

Then [VII. 20] $a = mc$, and $b = md$, where m is some integer.

Therefore $$a = cm, \quad b = dm,$$ [VII. 16]

and m is a common measure of a, b, though these are prime to one another. which is impossible. [VII. Def. 12]

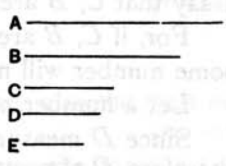
Thus the least numbers in the ratio of a to b cannot be less than a, b themselves.

Where I have quoted VII. 16 Heiberg regards the reference as being to VII. 15. I think the phraseology of the text combined with that of Def. 15 suggests the former rather than the latter.

PROPOSITION 22.

The least numbers of those which have the same ratio with them are prime to one another.

Let A, B be the least numbers of those which have the same ratio with them;



I say that A, B are prime to one another.

For, if they are not prime to one another, some number will measure them.

Let some number measure them, and let it be C.

And, as many times as C measures A, so many units let there be in D,
and, as many times as C measures B, so many units let there be in E

Since C measures A according to the units in D, therefore C by multiplying D has made A. [VII. Def. 15]

For the same reason also
C by multiplying E has made B.

Thus the number C by multiplying the two numbers D, E has made A, B;
therefore, as D is to E, so is A to B; [VII. 17]
therefore D, E are in the same ratio with A, B, being less than they: which is impossible.

Therefore no number will measure the numbers A, B.

Therefore A, B are prime to one another.

Q. E. D.

If $a : b$ is "in its lowest terms," a, b are prime to one another.

Again the proof is indirect.

If a, b are not prime to one another, they have some common measure c, and

$$a = mc, \quad b = nc.$$

Therefore $m : n = a : b$. [VII. 17 or 18]

But m, n are less than a, b respectively, so that $a : b$ is not in its lowest terms: which is contrary to the hypothesis.

Therefore etc.

Proposition 23.

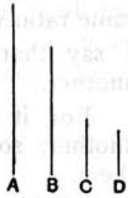
If two numbers be prime to one another, the number which measures the one of them will be prime to the remaining number.

Let A, B be two numbers prime to one another, and let any number C measure A;
I say that C, B are also prime to one another.

For, if C, B are not prime to one another,
some number will measure C, B.

Let a number measure them, and let it be D.

Since D measures C, and C measures A,
therefore D also measures A.



But it also measures B;

therefore D measures A, B which are prime to one another: which is impossible. [VII. Def. 12]

Therefore no number will measure the numbers C, B.

Therefore C, B are prime to one another.

Q. E. D.

If a, mb are prime to one another, b is prime to a. For, if not, some number d will measure both a and b, and therefore both a and mb: which is contrary to the hypothesis.

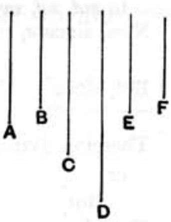
Therefore etc.

PROPOSITION 24.

If two numbers be prime to any number, their product also will be prime to the same.

For let the two numbers A, B be prime to any number C, and let A by multiplying B make D;

I say that C, D are prime to one another.



For, if C, D are not prime to one another, some number will measure C, D.

Let a number measure them, and let it be E.

Now, since C, A are prime to one another, and a certain number E measures C, therefore A, E are prime to one another. [VII. 23]

As many times, then, as E measures D, so many units let there be in F;

therefore F also measures D according to the units in E. [VII. 16]

Therefore E by multiplying F has made D. [VII. Def. 15]

But, further, A by multiplying B has also made D;

therefore the product of E, F is equal to the product of A, B.

But, if the product of the extremes be equal to that of the means, the four numbers are proportional; [VII. 19]

therefore, as E is to A, so is B to F.

But A, E are prime to one another,

numbers which are prime to one another are also the least of those which have the same ratio, [VII. 21]

and the least numbers of those which have the same ratio with them measure those which have the same ratio the same

number of times, the greater the greater and the less the less, that is, the antecedent the antecedent and the consequent the consequent; [VII. 20]
therefore E measures B.

But it also measures C;
therefore E measures B, C which are prime to one another: which is impossible. [VII. Def. 12]

Therefore no number will measure the numbers C, D.

Therefore C, D are prime to one another.

Q. E. D.

1. **their product.** ὁ ἐξ αὐτῶν γενόμενος, literally "the (number) produced from them," will henceforth be translated as "their product."

If a, b are both prime to c, then ab, c are prime to one another.

The proof is again by *reductio ad absurdum*.

If ab, c are not prime to one another, let them be measured by d and be equal to md, nd, say, respectively.

Now, since a, c are prime to one another and d measures c,

a, d are prime to one another. [VII. 23]

But, since $ab = md$,

$d : a = b : m$. [VII. 19]

Therefore [VII. 20] d measures b,

or $b = pd$, say.

But $c = nd$.

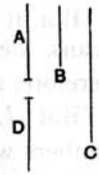
Therefore d measures both b and c, which are therefore not prime to one another: which is impossible.

Therefore etc.

Proposition 25.

If two numbers be prime to one another, the product of one of them into itself will be prime to the remaining one.

Let A, B be two numbers prime to one another, and let A by multiplying itself make C:
I say that B, C are prime to one another.



For let D be made equal to A.

Since A, B are prime to one another, and A is equal to D,
therefore D, B are also prime to one another.

Therefore each of the two numbers D, A is prime to B;
therefore the product of D, A will also be prime to B. [VII. 24]

But the number which is the product of D, A is C.

Therefore C, B are prime to one another. Q. E. D.

1. the product of one of them into itself. The Greek, ὁ ἐκ τοῦ ἑνὸς αὐτῶν γενόμενος, literally "the number produced from the one of them," leaves "multiplied into itself" to be understood.

If a, b are prime to one another,

a^2 is prime to b.

Euclid takes d equal to a, so that d, a are both prime to b.

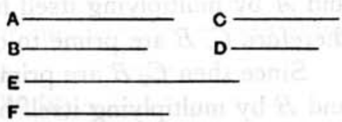
Hence, by VII. 24, da, i.e. a^2, is prime to b.

The proposition is a particular case of the preceding proposition; and the method of proof is by substitution of different numbers in the result of that proposition.

Proposition 26.

If two numbers be prime to two numbers, both to each, their products also will be prime to one another.

For let the two numbers A, B be prime to the two numbers C, D; both to each, and let A by multiplying B make E, and let C by multiplying D make F;



I say that E, F are prime to one another.

For, since each of the numbers A, B is prime to C, therefore the product of A, B will also be prime to C. [VII. 24]

But the product of A, B is E;

therefore E, C are prime to one another.

For the same reason

E, D are also prime to one another.

Therefore each of the numbers C, D is prime to E.

Therefore the product of C, D will also be prime to E. [VII. 24]

But the product of C, D is F.

Therefore E, F are prime to one another. Q. E. D.

If both a and b are prime to each of two numbers c, d, then ab, cd will be prime to one another.

Since a, b are both prime to c,

ab, c are prime to one another. [VII. 24]

Similarly ab, d are prime to one another.

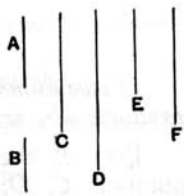
Therefore c, d are both prime to ab,

and so therefore is cd. [VII. 24]

Proposition 27.

If two numbers be prime to one another, and each by multiplying itself make a certain number, the products will be prime to one another; and, if the original numbers by multiplying the products make certain numbers, the latter will also be prime to one another [and this is always the case with the extremes].

Let A, B be two numbers prime to one another, let A by multiplying itself make C, and by multiplying C make D,
and let B by multiplying itself make E, and by multiplying E make F;
I say that both C, E and D, F are prime to one another.



For, since A, B are prime to one another, and A by multiplying itself has made C, therefore C, B are prime to one another. [VII. 25]

Since then C, B are prime to one another, and B by multiplying itself has made E, therefore C, E are prime to one another. [*id.*]

Again, since A, B are prime to one another, and B by multiplying itself has made E, therefore A, E are prime to one another. [*id.*]

Since then the two numbers A, C are prime to the two numbers B, E, both to each, therefore also the product of A, C is prime to the product of B, E. [VII. 26]

And the product of A, C is D, and the product of B, E is F.

Therefore D, F are prime to one another.

Q. E. D.

If a, b are prime to one another, so are a^2, b^2 and so are a^3, b^3; and, generally, a^n, b^n are prime to one another.

The words in the enunciation which assert the truth of the proposition for *any* powers are suspected and bracketed by Heiberg because (1) in περὶ τοὺς ἄκρους the use of ἄκροι is peculiar, for it can only mean "the last products," and (2) the words have nothing corresponding to them in the proof, much less is the generalisation proved. Campanus omits the words in the enunciation, though he adds to the proof a remark that the proposition is true of any, the same or different, powers of a, b. Heiberg concludes that the words are an interpolation of date earlier than Theon.

Euclid's proof amounts to this.

Since a, b are prime to one another, so are a^2, b [VII. 25], and therefore also a^2, b^2. [VII. 25]

Similarly [VII. 25] a, b^2 are prime to one another.

Therefore a, a^2 and b, b^2 satisfy the description in the enunciation of VII. 26.

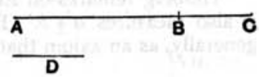
Hence a^3, b^3 are prime to one another.

PROPOSITION 28.

If two numbers be prime to one another, the sum will also be prime to each of them; and, if the sum of two numbers be prime to any one of them, the original numbers will also be prime to one another.

For let two numbers AB, BC prime to one another be added;

I say that the sum AC is also prime to each of the numbers AB, BC.



For, if CA, AB are not prime to one another,

some number will measure CA, AB.

Let a number measure them, and let it be D.

Since then D measures CA, AB,

therefore it will also measure the remainder BC.

But it also measures BA;

therefore D measures AB, BC which are prime to one another: which is impossible. [VII. Def. 12]

Therefore no number will measure the numbers CA, AB; therefore CA, AB are prime to one another.

For the same reason

AC, CB are also prime to one another.

Therefore CA is prime to each of the numbers AB, BC.

Again, let CA, AB be prime to one another;

I say that AB, BC are also prime to one another.

For, if AB, BC are not prime to one another,

some number will measure AB, BC.

Let a number measure them, and let it be D.

Now, since D measures each of the numbers AB, BC, it will also measure the whole CA.

But it also measures AB;

therefore D measures CA, AB which are prime to one another: which is impossible. [VII. Def. 12]

Therefore no number will measure the numbers AB, BC.

Therefore AB, BC are prime to one another.

Q. E. D.

If a, b are prime to one another, $a+b$ will be prime to both a and b; and conversely.

For suppose $(a+b)$, a are not prime to one another. They must then have some common measure d.

Therefore d also divides the difference $(a+b)-a$, or b, as well as a; and therefore a, b are not prime to one another: which is contrary to the hypothesis.

Therefore $\quad a+b$ is prime to a.

Similarly $\quad a+b$ is prime to b.

The converse is proved in the same way.

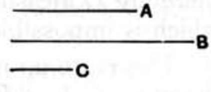
Heiberg remarks on Euclid's assumption that, if c measures both a and b, it also measures $a \pm b$. But it has already (VII. 1, 2) been assumed, more generally, as an axiom that, in the case supposed, c measures $a \pm pb$.

Proposition 29.

Any prime number is prime to any number which it does not measure.

Let A be a prime number, and let it not measure B;

I say that B, A are prime to one another.



For, if B, A are not prime to one another,

some number will measure them.

Let C measure them.

Since C measures B,

and A does not measure B,

therefore C is not the same with A.

Now, since C measures B, A,

therefore it also measures A which is prime, though it is not the same with it:

which is impossible.

Therefore no number will measure B, A.

Therefore A, B are prime to one another.

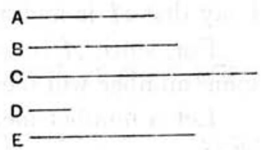
Q. E. D.

If a is prime and does not measure b, then a, b are prime to one another. The proof is self-evident.

PROPOSITION 30.

If two numbers by multiplying one another make some number, and any prime number measure the product, it will also measure one of the original numbers.

For let the two numbers A, B by multiplying one another make C, and let any prime number D measure C;



I say that D measures one of the numbers A, B.

For let it not measure A.

Now D is prime;

therefore A, D are prime to one another. [VII. 29]

And, as many times as D measures C, so many units let there be in E.

Since then D measures C according to the units in E, therefore D by multiplying E has made C. [VII. Def. 15]

Further, A by multiplying B has also made C;

therefore the product of D, E is equal to the product of A, B.

Therefore, as D is to A, so is B to E. [VII. 19]

But D, A are prime to one another,

primes are also least, [VII. 21]

and the least measure the numbers which have the same ratio the same number of times, the greater the greater and the less the less, that is, the antecedent the antecedent and the consequent the consequent; [VII. 20]

therefore D measures B.

Similarly we can also show that, if D do not measure B, it will measure A.

Therefore D measures one of the numbers A, B.

Q. E. D.

If c, a prime number, measure ab, c will measure either a or b.
Suppose c does not measure a.
Therefore c, a are prime to one another. [VII. 29]
Suppose $ab = mc$.
Therefore $c : a = b : m$. [VII. 19]
Hence [VII. 20, 21] c measures b.
Similarly, if c does not measure b, it measures a.
Therefore it measures one or other of the two numbers a, b.

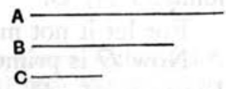
Proposition 31.

Any composite number is measured by some prime number.

Let A be a composite number;

I say that A is measured by some prime number.

For, since A is composite, some number will measure it.



Let a number measure it, and let it be B.

Now, if B is prime, what was enjoined will have been done.

But if it is composite, some number will measure it.

Let a number measure it, and let it be C.

Then, since C measures B, and B measures A, therefore C also measures A.

And, if C is prime, what was enjoined will have been done.

But if it is composite, some number will measure it.

Thus, if the investigation be continued in this way, some prime number will be found which will measure the number before it, which will also measure A.

For, if it is not found, an infinite series of numbers will measure the number A, each of which is less than the other: which is impossible in numbers.

Therefore some prime number will be found which will measure the one before it, which will also measure A.

Therefore any composite number is measured by some prime number.

8. **if B is prime, what was enjoined will have been done,** i.e. the implied *problem* of finding a prime number which measures A.

18. **some prime number will be found which will measure.** In the Greek the sentence stops here, but it is necessary to add the words "the number before it, which will also measure A," which are found a few lines further down. It is possible that the words may have fallen out of P here by a simple mistake due to ὁμοιοτέλευτον (Heiberg).

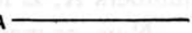
Heiberg relegates to the Appendix an alternative proof of this proposition, to the following effect. Since A is composite, some number will measure it. Let B be the *least* such number. I say that B is prime. For, if not, B is composite, and some number will measure it, say C; so that C is less than B. But, since C measures B, and B measures A, C must measure A. And C is less than B: which is contrary to the hypothesis.

Proposition 32.

Any number either is prime or is measured by some prime number.

Let A be a number;

I say that A either is prime or is measured by some prime number.



If now A is prime, that which was enjoined will have been done.

But if it is composite, some prime number will measure it. [VII. 31]

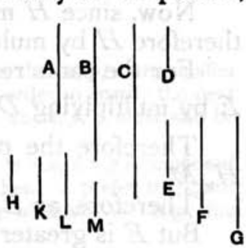
Therefore any number either is prime or is measured by some prime number.

Q. E. D.

Proposition 33.

Given as many numbers as we please, to find the least of those which have the same ratio with them.

Let A, B, C be the given numbers, as many as we please; thus it is required to find the least of those which have the same ratio with A, B, C.



A, B, C are either prime to one another or not.

Now, if A, B, C are prime to one another, they are the least of those which have the same ratio with them. [VII. 21]

But, if not, let D the greatest common measure of A, B, C be taken, [VII. 3] and, as many times as D measures the numbers A, B, C respectively, so many units let there be in the numbers E, F, G respectively.

Therefore the numbers E, F, G measure the numbers A, B, C respectively according to the units in D. [VII. 16]

Therefore E, F, G measure A, B, C the same number of times;

therefore E, F, G are in the same ratio with A, B, C. [VII. Def. 20]

I say next that they are the least that are in that ratio.

For, if E, F, G are not the least of those which have the same ratio with A, B, C,

there will be numbers less than E, F, G which are in the same ratio with A, B, C.

Let them be H, K, L;

therefore H measures A the same number of times that the numbers K, L measure the numbers B, C respectively.

Now, as many times as H measures A, so many units let there be in M;

therefore the numbers K, L also measure the numbers B, C respectively according to the units in M.

And, since H measures A according to the units in M, therefore M also measures A according to the units in H. [VII. 16]

For the same reason

M also measures the numbers B, C according to the units in the numbers K, L respectively;

Therefore M measures A, B, C.

Now, since H measures A according to the units in M, therefore H by multiplying M has made A. [VII. Def. 15]

For the same reason also

E by multiplying D has made A.

Therefore the product of E, D is equal to the product of H, M.

Therefore, as E is to H, so is M to D. [VII. 19]

But E is greater than H;

therefore M is also greater than D.

And it measures A, B, C:

which is impossible, for by hypothesis D is the greatest common measure of A, B, C.

Therefore there cannot be any numbers less than E, F, G which are in the same ratio with A, B, C.

Therefore E, F, G are the least of those which have the same ratio with A, B, C.

Q. E. D.

17. **the numbers E, F, G measure the numbers A, B, C respectively,** literally (as usual) "each of the numbers E, F, G measures each of the numbers A, B, C."

Given any numbers $a, b, c, \ldots$, to find the least numbers that are in the same ratio.

Euclid's method is the obvious one, and the result is verified by *reductio ad absurdum*.

We will, like Euclid, take three numbers only, a, b, c.

Let g, their greatest common measure, be found [VII. 3], and suppose that

$$a = mg, \text{ i.e. } gm, \qquad \text{[VII. 16]}$$
$$b = ng, \text{ i.e. } gn,$$
$$c = pg, \text{ i.e. } gp.$$

It follows, by VII. Def. 20, that

$$m : n : p = a : b : c.$$

m, n, p shall be the numbers required.

For, if not, let x, y, z be the least numbers in the same ratio as a, b, c, being less than m, n, p.

Therefore $\quad a = kx$ (or xk, VII. 16),

$b = ky$ (or yk),

$c = kz$ (or zk),

where k is some integer. [VII. 20]

Thus $\quad mg = a = xk$.

Therefore $\quad m : x = k : g$. [VII. 19]

And $m > x$; therefore $k > g$.

Since then k measures a, b, c, it follows that g is not the greatest common measure: which contradicts the hypothesis.

Therefore etc.

It is to be observed that Euclid merely supposes that x, y, z are smaller numbers than m, n, p in the ratio of a, b, c; but, in order to justify the next inference, which apparently can only depend on VII. 20, x, y, z must also be assumed to be the *least* numbers in the ratio of a, b, c.

The inference from the last proportion that, since $m > x$, $k > g$ is supposed by Heiberg to depend upon VII. 13 and V. 14 together. I prefer to regard Euclid as making the inference quite independently of Book V. E.g., the proportion could just as well be written

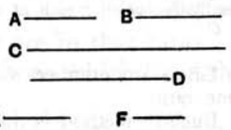
$$x : m = g : k,$$

when the definition of proportion in Book VII. (Def. 20) gives all that we want, since, whatever proper fraction x is of m, the same proper fraction is g of k.

PROPOSITION 34.

Given two numbers, to find the least number which they measure.

Let A, B be the two given numbers;
thus it is required to find the least number which they measure.



Now A, B are either prime to one another or not.

First, let A, B be prime to one another, and let A by multiplying B make C;
therefore also B by multiplying A has made C. [VII. 16]

Therefore A, B measure C.

I say next that it is also the least number they measure.

For, if not, A, B will measure some number which is less than C.

Let them measure D.

Then, as many times as A measures D, so many units let there be in E,
and, as many times as B measures D, so many units let there be in F;
therefore A by multiplying E has made D,
and B by multiplying F has made D; [VII. Def. 15]
therefore the product of A, E is equal to the product of B, F.

Therefore, as A is to B, so is F to E. [VII. 19]

But A, B are prime,
primes are also least, [VII. 21]
and the least measure the numbers which have the same ratio the same number of times, the greater the greater and the less the less; [VII. 20]
therefore B measures E, as consequent consequent.

And, since A by multiplying B, E has made C, D,
therefore, as B is to E, so is C to D. [VII. 17]

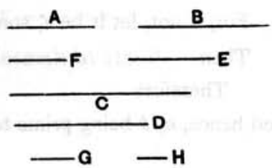
But B measures E;
therefore C also measures D, the greater the less:
which is impossible.

Therefore A, B do not measure any number less than C; therefore C is the least that is measured by A, B.

Next, let A, B not be prime to one another,

and let F, E, the least numbers of those which have the same ratio with A, B, be taken; [VII. 33]

therefore the product of A, E is equal to the product of B, F. [VII. 19]



And let A by multiplying E make C;

therefore also B by multiplying F has made C;

therefore A, B measure C.

I say next that it is also the least number that they measure.

For, if not, A, B will measure some number which is less than C.

Let them measure D.

And, as many times as A measures D, so many units let there be in G,

and, as many times as B measures D, so many units let there be in H.

Therefore A by multiplying G has made D,

and B by multiplying H has made D.

Therefore the product of A, G is equal to the product of B, H;

therefore, as A is to B, so is H to G. [VII. 19]

But, as A is to B, so is F to E.

Therefore also, as F is to E, so is H to G.

But F, E are least,

and the least measure the numbers which have the same ratio the same number of times, the greater the greater and the less the less; [VII. 20]

therefore E measures G.

And, since A by multiplying E, G has made C, D,

therefore, as E is to G, so is C to D. [VII. 17]

But E measures G;

therefore C also measures D, the greater the less:

which is impossible.

Therefore A, B will not measure any number which is less than C.

Therefore C is the least that is measured by A, B.

Q. E. D.

This is the problem of finding the *least common multiple* of two numbers, as a, b.

I. If a, b be prime to one another, the L.C.M. is ab.

For, if not, let it be d, some number less than ab.

Then $d = ma = nb$, where m, n are integers.

Therefore $a : b = n : m$, [VII. 19]

and hence, a, b being prime to one another,

b measures m. [VII. 20, 21]

But $b : m = ab : am$ [VII. 17]

$= ab : d$.

Therefore ab measures d: which is impossible.

II. If a, b be not prime to one another, find the numbers which are the least of those having the ratio of a to b, say m, n; [VII. 33]

then $a : b = m : n$,

and $an = bm$ $(= c$, say$)$; [VII. 19]

c is then the L.C.M.

For, if not, let it be d $(< c)$, so that

$ap = bq = d$, where p, q are integers.

Then $a : b = q : p$, [VII. 19]

whence $m : n = q : p$,

so that n measures p. [VII. 20, 21]

And $n : p = an : ap = c : d$,

so that c measures d:

which is impossible.

Therefore etc.

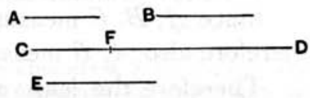
By VII. 33, $\left.\begin{aligned} m &= \frac{a}{g} \\ n &= \frac{b}{g} \end{aligned}\right\}$, where g is the G.C.M. of a, b.

Hence the L.C.M. is $\dfrac{ab}{g}$.

PROPOSITION 35.

If two numbers measure any number, the least number measured by them will also measure the same.

For let the two numbers A, B measure any number CD, and let E be the least that they measure;



I say that E also measures CD.

For, if E does not measure CD, let E, measuring DF, leave CF less than itself.

Now, since A, B measure E, and E measures DF, therefore A, B will also measure DF.

But they also measure the whole CD; therefore they will also measure the remainder CF which is less than E: which is impossible.

Therefore E cannot fail to measure CD; therefore it measures it.

Q. E. D.

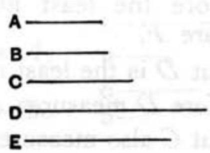
The *least* common multiple of any two numbers must measure any other common multiple.

The proof is obvious, depending on the fact that, if any number divides a and b, it also divides $a-pb$.

PROPOSITION 36.

Given three numbers, to find the least number which they measure.

Let A, B, C be the three given numbers; thus it is required to find the least number which they measure.



Let D, the least number measured by the two numbers A, B, be taken. [VII. 34]

Then C either measures, or does not measure, D.

First, let it measure it.

But A, B also measure D;
therefore A, B, C measure D.

I say next that it is also the least that they measure.

For, if not, A, B, C will measure some number which is less than D.

Let them measure E.

Since A, B, C measure E,
therefore also A, B measure E.

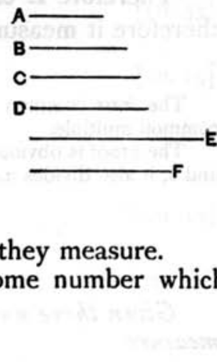
Therefore the least number measured by A, B will also measure E. [VII. 35]

But D is the least number measured by A, B;
therefore D will measure E, the greater the less:
which is impossible.

Therefore A, B, C will not measure any number which is less than D;

therefore D is the least that A, B, C measure.

Again, let C not measure D,
and let E, the least number measured by C, D, be taken. [VII. 34]



Since A, B measure D,
and D measures E,
therefore also A, B measure E.

But C also measures E;
therefore also A, B, C measure E.

I say next that it is also the least that they measure.

For, if not, A, B, C will measure some number which is less than E.

Let them measure F.

Since A, B, C measure F,
therefore also A, B measure F;
therefore the least number measured by A, B will also measure F. [VII. 35]

But D is the least number measured by A, B;
therefore D measures F.

But C also measures F;
therefore D, C measure F,
so that the least number measured by D, C will also measure F.

But E is the least number measured by C, D; therefore E measures F, the greater the less: which is impossible.

Therefore A, B, C will not measure any number which is less than E.

Therefore E is the least that is measured by A, B, C.

Q. E. D.

Euclid's rule for finding the L.C.M. of *three* numbers a, b, c is the rule with which we are familiar. The L.C.M. of a, b is first found, say d, and then the L.C.M. of d and c is found.

Euclid distinguishes the cases (1) in which c measures d, (2) in which c does not measure d. We need only reproduce the proof of the general case (2). The method is that of *reductio ad absurdum*.

Let e be the L.C.M. of d, c.

Since a, b both measure d, and d measures e,

a, b both measure e.

So does c.

Therefore e is *some* common multiple of a, b, c.

If it is not the *least*, let f be the L.C.M.

Now a, b both measure f;

therefore d, their L.C.M., also measures f. [VII. 35]

Thus d, c both measure f;

therefore e, their L.C.M., measures f: [VII. 35]

which is impossible, since $f < e$.

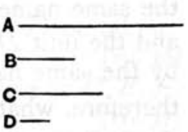
Therefore etc.

The process can be continued *ad libitum*, so that we can find the L.C.M., not only of three, but of as many numbers as we please.

Proposition 37.

If a number be measured by any number, the number which is measured will have a part called by the same name as the measuring number.

For let the number A be measured by any number B; I say that A has a part called by the same name as B.



For, as many times as B measures A, so many units let there be in C.

Since B measures A according to the units in C,

and the unit D also measures the number C according to the units in it, therefore the unit D measures the number C the same number of times as B measures A.

Therefore, alternately, the unit D measures the number B the same number of times as C measures A; [VII. 15] therefore, whatever part the unit D is of the number B, the same part is C of A also.

But the unit D is a part of the number B called by the same name as it;
therefore C is also a part of A called by the same name as B, so that A has a part C which is called by the same name as B.

Q. E. D.

If b measures a, then $\frac{1}{b}$ th of a is a whole number.

Let $a = m \cdot b$.
Now $m = m \cdot 1$.

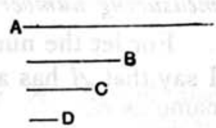
Thus $1, m, b, a$ satisfy the enunciation of VII. 15;
therefore m measures a the same number of times that 1 measures b.

But 1 is $\frac{1}{b}$ th part of b;
therefore m is $\frac{1}{b}$ th part of a.

PROPOSITION 38.

If a number have any part whatever, it will be measured by a number called by the same name as the part.

For let the number A have any part whatever, B, and let C be a number called by the same name as the part B;
I say that C measures A.



For, since B is a part of A called by the same name as C,
and the unit D is also a part of C called by the same name as it,
therefore, whatever part the unit D is of the number C, the same part is B of A also;
therefore the unit D measures the number C the same number of times that B measures A.

Therefore, alternately, the unit D measures the number B the same number of times that C measures A. [VII. 15]

Therefore C measures A.

Q. E. D.

This proposition is practically a restatement of the preceding proposition. It asserts that, if b is $\frac{1}{m}$ th part of a,

i.e., if $$b = \frac{1}{m} a,$$

then $$m \text{ measures } a.$$

We have $$b = \frac{1}{m} a,$$

and $$1 = \frac{1}{m} m.$$

Therefore 1, m, b, a, satisfy the enunciation of VII. 15, and therefore m measures a the same number of times as 1 measures b, or

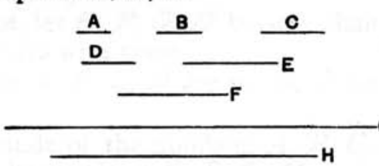
$$m = \frac{1}{b} a.$$

Proposition 39.

To find the number which is the least that will have given parts.

Let A, B, C be the given parts;

thus it is required to find the number which is the least that will have the parts A, B, C.



Let D, E, F be numbers called by the same name as the parts A, B, C,

and let G, the least number measured by D, E, F, be taken. [VII. 36]

Therefore G has parts called by the same name as D, E, F. [VII. 37]

But A, B, C are parts called by the same name as D, E, F;

therefore G has the parts A, B, C.

I say next that it is also the least number that has.

For, if not, there will be some number less than G which will have the parts A, B, C.

Let it be H.

Since H has the parts A, B, C,

therefore H will be measured by numbers called by the same name as the parts A, B, C. [VII. 38]

But D, E, F are numbers called by the same name as the parts A, B, C;

therefore H is measured by D, E, F.

And it is less than G: which is impossible.

Therefore there will be no number less than G that will have the parts A, B, C.

Q. E. D.

This again is practically a restatement in another form of the problem of finding the L.C.M.

To find a number which has $\frac{1}{a}$th, $\frac{1}{b}$th and $\frac{1}{c}$th parts.

Let d be the L.C.M. of a, b, c.

Thus d has $\frac{1}{a}$th, $\frac{1}{b}$th and $\frac{1}{c}$th parts. [VII. 37]

If it is not the least number which has, let the least such number be e.

Then, since e has those parts,

e is measured by a, b, c; and $e < d$:

which is impossible.

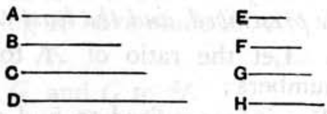
BOOK VIII.

Proposition 1.

If there be as many numbers as we please in continued proportion, and the extremes of them be prime to one another, the numbers are the least of those which have the same ratio with them.

Let there be as many numbers as we please, A, B, C, D, in continued proportion,

and let the extremes of them A, D be prime to one another;



I say that A, B, C, D are the least of those which have the same ratio with them.

For, if not, let E, F, G, H be less than A, B, C, D, and in the same ratio with them.

Now, since A, B, C, D are in the same ratio with E, F, G, H,

and the multitude of the numbers A, B, C, D is equal to the multitude of the numbers E, F, G, H,

therefore, *ex aequali*,

as A is to D, so is E to H. [VII. 14]

But A, D are prime,

primes are also least, [VII. 21]

and the least numbers measure those which have the same ratio the same number of times, the greater the greater and the less the less, that is, the antecedent the antecedent and the consequent the consequent. [VII. 20]

Therefore A measures E, the greater the less: which is impossible.

Therefore E, F, G, H which are less than A, B, C, D are not in the same ratio with them.

Therefore A, B, C, D are the least of those which have the same ratio with them.

Q. E. D.

What we call a *geometrical progression* is with Euclid a series of terms "in continued proportion" (ἑξῆς ἀνάλογον).

This proposition proves that, if a, b, c, ... k are a series of numbers in geometrical progression, and if a, k are prime to one another, the series is in the lowest terms possible with the same common ratio.

The proof is in *form* by *reductio ad absurdum*. We should no doubt desert this *form* while retaining the substance. If a', b', c', ... k' be *any* other series of numbers in G.P. with the same common ratio as before, we have, *ex aequali*,

$$a : k = a' : k', \qquad \text{[VII. 14]}$$

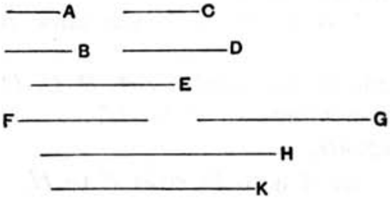
whence, since a, k are prime to one another, a, k measure a', k' respectively, so that a', k' are *greater* than a, k respectively.

PROPOSITION 2.

To find numbers in continued proportion, as many as may be prescribed, and the least that are in a given ratio.

Let the ratio of A to B be the given ratio in least numbers;

thus it is required to find numbers in continued proportion, as many as may be prescribed, and the least that are in the ratio of A to B.



Let four be prescribed;

let A by multiplying itself make C, and by multiplying B let it make D;

let B by multiplying itself make E;

further, let A by multiplying C, D, E make F, G, H,

and let B by multiplying E make K.

Now, since A by multiplying itself has made C, and by multiplying B has made D, therefore, as A is to B, so is C to D. [VII. 17]

Again, since A by multiplying B has made D, and B by multiplying itself has made E, therefore the numbers A, B by multiplying B have made the numbers D, E respectively.

Therefore, as A is to B, so is D to E. [VII. 18]

But, as A is to B, so is C to D; therefore also, as C is to D, so is D to E.

And, since A by multiplying C, D has made F, G, therefore, as C is to D, so is F to G. [VII. 17]

But, as C is to D, so was A to B; therefore also, as A is to B, so is F to G.

Again, since A by multiplying D, E has made G, H, therefore, as D is to E, so is G to H. [VII. 17]

But, as D is to E, so is A to B.

Therefore also, as A is to B, so is G to H.

And, since A, B by multiplying E have made H, K, therefore, as A is to B, so is H to K. [VII. 18]

But, as A is to B, so is F to G, and G to H.

Therefore also, as F is to G, so is G to H, and H to K; therefore C, D, E, and F, G, H, K are proportional in the ratio of A to B.

I say next that they are the least numbers that are so.

For, since A, B are the least of those which have the same ratio with them, and the least of those which have the same ratio are prime to one another, [VII. 22] therefore A, B are prime to one another.

And the numbers A, B by multiplying themselves respectively have made the numbers C, E, and by multiplying the numbers C, E respectively have made the numbers F, K; therefore C, E and F, K are prime to one another respectively. [VII. 27]

But, if there be as many numbers as we please in continued proportion, and the extremes of them be prime to one another, they are the least of those which have the same ratio with them. [VIII. 1]

Therefore C, D, E and F, G, H, K are the least of those which have the same ratio with A, B. Q. E. D.

PORISM. From this it is manifest that, if three numbers in continued proportion be the least of those which have the same ratio with them, the extremes of them are squares, and, if four numbers, cubes.

To find a series of numbers in geometrical progression and in the least terms which have a given common ratio (understanding by that term *the ratio of one term to the next*).

Reduce the given ratio to its lowest terms, say, $a : b$. (This can be done by VII. 33.)

Then $a^n, a^{n-1}b, a^{n-2}b^2, \dots a^2b^{n-2}, ab^{n-1}, b^n$

is the required series of numbers if $(n+1)$ terms are required.

That this is a series of terms with the given common ratio is clear from VII. 17, 18.

That the G.P. is in the smallest terms possible is proved thus.

a, b are prime to one another, since the ratio $a : b$ is in its lowest terms. [VII. 22]

Therefore a^2, b^2 are prime to one another; so are a^3, b^3 and, generally, a^n, b^n. [VII. 27]

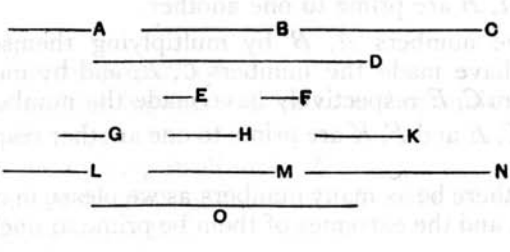
Whence the G.P. is in the smallest possible terms, by VIII. 1.

The Porism observes that, if there are n terms in the series, the extremes are $(n-1)$th powers.

PROPOSITION 3.

If as many numbers as we please in continued proportion be the least of those which have the same ratio with them, the extremes of them are prime to one another.

Let as many numbers as we please, A, B, C, D, in continued proportion be the least of those which have the same ratio with them;



I say that the extremes of them A, D are prime to one another.

For let two numbers E, F, the least that are in the ratio of A, B, C, D, be taken, [VII. 33]

then three others G, H, K with the same property;

and others, more by one continually, [VIII. 2]

until the multitude taken becomes equal to the multitude of the numbers A, B, C, D.

Let them be taken, and let them be L, M, N, O.

Now, since E, F are the least of those which have the same ratio with them, they are prime to one another. [VII. 22]

And, since the numbers E, F by multiplying themselves respectively have made the numbers G, K, and by multiplying the numbers G, K respectively have made the numbers L, O, [VIII. 2, Por.]

therefore both G, K and L, O are prime to one another. [VII. 27]

And, since A, B, C, D are the least of those which have the same ratio with them,

while L, M, N, O are the least that are in the same ratio with A, B, C, D,

and the multitude of the numbers A, B, C, D is equal to the multitude of the numbers L, M, N, O,

therefore the numbers A, B, C, D are equal to the numbers L, M, N, O respectively;

therefore A is equal to L, and D to O.

And L, O are prime to one another.

Therefore A, D are also prime to one another.

Q. E. D.

The proof consists in merely equating the given numbers to the terms of a series found in the manner of VIII. 2.

If $a, b, c, \dots k$ (n terms) be a geometrical progression in the lowest terms having a given common ratio, the terms must respectively be of the form

$$\alpha^{n-1},\quad \alpha^{n-2}\beta, \dots \alpha^2\beta^{n-3},\quad \alpha\beta^{n-2},\quad \beta^{n-1}$$

found by VIII. 2, where $\alpha : \beta$ is the ratio $a : b$ expressed in its lowest terms, so that α, β are prime to one another [VII. 22], and hence α^{n-1}, β^{n-1} are prime to one another [VII. 27].

But the two series must be the same, so that

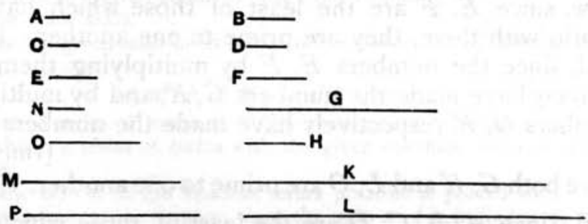
$$a = \alpha^{n-1},\quad b = \beta^{n-1}$$

Proposition 4.

Given as many ratios as we please in least numbers, to find numbers in continued proportion which are the least in the given ratios.

Let the given ratios in least numbers be that of A to B, that of C to D, and that of E to F;

thus it is required to find numbers in continued proportion which are the least that are in the ratio of A to B, in the ratio of C to D, and in the ratio of E to F.



Let G, the least number measured by B, C, be taken. [VII. 34]

And, as many times as B measures G, so many times also let A measure H,

and, as many times as C measures G, so many times also let D measure K.

Now E either measures or does not measure K.

First, let it measure it.

And, as many times as E measures K, so many times let F measure L also.

Now, since A measures H the same number of times that B measures G,

therefore, as A is to B, so is H to G. [VII. Def. 20, VII. 13]

For the same reason also,

as C is to D, so is G to K,

and further, as E is to F, so is K to L;

therefore H, G, K, L are continuously proportional in the ratio of A to B, in the ratio of C to D, and in the ratio of E to F.

I say next that they are also the least that have this property.

For, if H, G, K, L are not the least numbers continuously proportional in the ratios of A to B, of C to D, and of E to F, let them be N, O, M, P.

Then since, as A is to B, so is N to O,

while A, B are least,

and the least numbers measure those which have the same ratio the same number of times, the greater the greater and the less the less, that is, the antecedent the antecedent and the consequent the consequent;

therefore B measures O. [VII. 20]

For the same reason

C also measures O;

therefore B, C measure O;

therefore the least number measured by B, C will also measure O. [VII. 35]

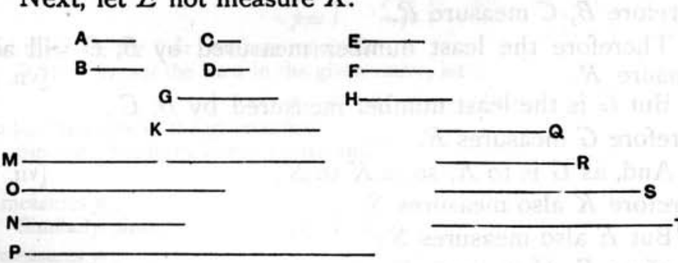
But G is the least number measured by B, C;

therefore G measures O, the greater the less:

which is impossible.

Therefore there will be no numbers less than H, G, K, L which are continuously in the ratio of A to B, of C to D, and of E to F.

Next, let E not measure K.



Let M, the least number measured by E, K, be taken.

And, as many times as K measures M, so many times let H, G measure N, O respectively,

and, as many times as E measures M, so many times let F measure P also.

Since H measures N the same number of times that G measures O,

therefore, as H is to G, so is N to O. [VII. 13 and Def. 20]

But, as H is to G, so is A to B; therefore also, as A is to B, so is N to O.

For the same reason also,

as C is to D, so is O to M.

Again, since E measures M the same number of times that F measures P, therefore, as E is to F, so is M to P; [VII. 13 and Def. 20] therefore N, O, M, P are continuously proportional in the ratios of A to B, of C to D, and of E to F.

I say next that they are also the least that are in the ratios $A:B$, $C:D$, $E:F$.

For, if not, there will be some numbers less than N, O, M, P continuously proportional in the ratios $A:B$, $C:D$, $E:F$.

Let them be Q, R, S, T.

Now since, as Q is to R, so is A to B, while A, B arc least, and the least numbers measure those which have the same ratio with them the same number of times, the antecedent the antecedent and the consequent the consequent, [VII. 20] therefore B measures R.

For the same reason C also measures R; therefore B, C measure R.

Therefore the least number measured by B, C will also measure R. [VII. 35]

But G is the least number measured by B, C; therefore G measures R.

And, as G is to R, so is K to S: [VII. 13] therefore K also measures S.

But E also measures S; therefore E, K measure S.

Therefore the least number measured by E, K will also measure S. [VII. 35]

But M is the least number measured by E, K; therefore M measures S, the greater the less: which is impossible.

Therefore there will not be any numbers less than N, O, M, P continuously proportional in the ratios of A to B, of C to D, and of E to F;

therefore N, O, M, P are the least numbers continuously proportional in the ratios $A:B$, $C:D$, $E:F$. Q. E. D.

69, 71, 99. **the ratios A : B, C : D, E : F.** This abbreviated expression is in the Greek οἱ ΑΒ, ΓΔ, ΕΖ λόγοι.

The term "in continued proportion" is here not used in its proper sense, since a geometrical progression is not meant, but a series of terms each of which bears to the succeeding term a *given*, but not the *same*, ratio.

The proposition furnishes a good example of the cumbrousness of the Greek method of dealing with non-determinate numbers. The proof in fact is not easy to follow without the help of modern symbolical notation. If this be used, the reasoning can be made clear enough.

Euclid takes *three* given ratios and therefore requires to find *four* numbers. We will leave out the simpler particular case which he puts first, that namely in which E accidentally measures K, the multiple of D found in the first few lines; and we will reproduce the general case with *three* ratios.

Let the ratios in their lowest terms be

$$a:b,\quad c:d,\quad e:f$$

Take l_1, the L.C.M. of b, c, and suppose that

$$l_1 = mb = nc.$$

Form the numbers $\quad ma,\ \left.\begin{matrix} mb \\ =nc \end{matrix}\right\},\ nd.$

These are in the ratios of a to b and of c to d respectively.

Next, let l_2 be the L.C.M. of nd, e, and let

$$l_2 = pnd = qe.$$

Now form the numbers

$$pma,\quad \left.\begin{matrix} pmb \\ =pnc \end{matrix}\right\},\quad \left.\begin{matrix} pnd \\ =qe \end{matrix}\right\},\quad qf,$$

and these are the four numbers required.

If they are *not* the least in the given ratios, let

$$x,\quad y,\quad z,\quad u$$

be less numbers in the given ratios.

Since $a:b$ is in its lowest terms, and

$$a:b = x:y,$$

b measures y.

Similarly, since $\quad c:d = y:z,$

c measures y.

Therefore l_1, the L.C.M. of b, c, measures y.

But $\quad l_1 : nd\,[= c:d] = y:z.$

Therefore nd measures z.

And, since $\quad e:f = z:u,$

e measures z.

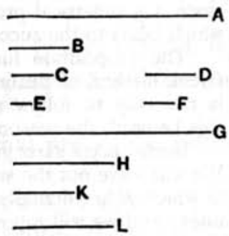
Therefore l_2, the L.C.M. of nd, e, measures z: which is impossible, since $z < l_2$ or pnd.

The step (line 86) inferring that $G:R = K:S$ is of course *alternando* from $G:K\,[= C:D] = R:S$.

It will be observed that VIII. 4 corresponds to the portion of VI. 23 which shows how to *compound* two ratios between straight lines.

PROPOSITION 5.

Plane numbers have to one another the ratio compounded of the ratios of their sides.



Let A, B be plane numbers, and let the numbers C, D be the sides of A, and E, F of B; I say that A has to B the ratio compounded of the ratios of the sides.

For, the ratios being given which C has to E and D to F, let the least numbers G, H, K that are continuously in the ratios $C:E$, $D:F$ be taken, so that,

as C is to E, so is G to H,

and, as D is to F, so is H to K. [VIII. 4]

And let D by multiplying E make L.

Now, since D by multiplying C has made A, and by multiplying E has made L,

therefore, as C is to E, so is A to L. [VII. 17]

But, as C is to E, so is G to H;

therefore also, as G is to H, so is A to L.

Again, since E by multiplying D has made L, and further by multiplying F has made B,

therefore, as D is to F, so is L to B. [VII. 17]

But, as D is to F, so is H to K;

therefore also, as H is to K, so is L to B.

But it was also proved that,

as G is to H, so is A to L;

therefore, *ex aequali*,

as G is to K, so is A to B. [VII. 14]

But G has to K the ratio compounded of the ratios of the sides;

therefore A also has to B the ratio compounded of the ratios of the sides.

Q. E. D.

1, 5, 29, 31. **compounded of the ratios of their sides.** As in VI. 23, the Greek has the less exact phrase, "compounded of their sides."

If $a = cd, \quad b = ef,$

then a has to b the ratio compounded of $c : e$ and $d : f$.

Take three numbers the least which are continuously in the given ratios.

If l is the L.C.M. of e, d and $l = me = nd$, the three numbers are

$$mc, \quad \left.\begin{matrix} me \\ = nd \end{matrix}\right\}, \quad nf. \qquad [\text{VIII. } 4]$$

Now $$dc : de = c : e \qquad [\text{VII. } 17]$$
$$= mc : me = mc : nd.$$

Also $$ed : ef = d : f \qquad [\text{VII. } 17]$$
$$= nd : nf.$$

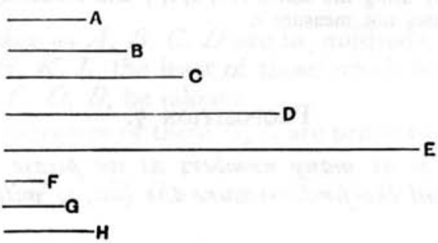
Therefore, *ex aequali*, $$cd : ef = mc : nf$$
$$= (\text{ratio compounded of } c : e \text{ and } d : f).$$

It will be seen that this proof follows exactly the method of VI. 23 for parallelograms.

PROPOSITION 6.

If there be as many numbers as we please in continued proportion, and the first do not measure the second, neither will any other measure any other.

Let there be as many numbers as we please, A, B, C, D, E, in continued proportion, and let A not measure B;
I say that neither will any other measure any other.



Now it is manifest that A, B, C, D, E do not measure one another in order; for A does not even measure B.

I say, then, that neither will any other measure any other.

For, if possible, let A measure C.

And, however many A, B, C are, let as many numbers F, G, H, the least of those which have the same ratio with A, B, C, be taken. [VII. 33]

Now, since F, G, H are in the same ratio with A, B, C, and the multitude of the numbers A, B, C is equal to the multitude of the numbers F, G, H,
therefore, *ex aequali*, as A is to C, so is F to H. [VII. 14]

And since, as A is to B, so is F to G,
while A does not measure B,
therefore neither does F measure G; [VII. Def. 20]
therefore F is not an unit, for the unit measures any number.

Now F, H are prime to one another. [VIII. 3]

And, as F is to H, so is A to C;
therefore neither does A measure C.

Similarly we can prove that neither will any other measure any other.

Q. E. D.

Let $a, b, c \ldots k$ be a geometrical progression in which a does not measure b.

Suppose, if possible, that a measures some term of the series, as f.

Take x, y, z, u, v, w the *least* numbers in the ratio a, b, c, d, e, f.

Since $$x : y = a : b,$$
and a does not measure b,
x does not measure y; therefore x cannot be unity.

And, *ex aequali*, $$x : w = a : f.$$

Now x, w are prime to one another. [VIII. 3]

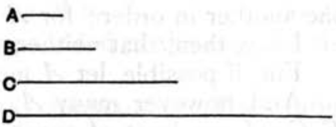
Therefore a does not measure f.

We can of course prove that an intermediate term, as b, does not measure a later term f by using the series b, c, d, e, f and remembering that, since $b : c = a : b$, b does not measure c.

PROPOSITION 7.

If there be as many numbers as we please in continued proportion, and the first measure the last, it will measure the second also.

Let there be as many numbers as we please, A, B, C, D, in continued proportion; and let A measure D;
I say that A also measures B.



For, if A does not measure B, neither will any other of the numbers measure any other. [VIII. 6]

But A measures D.

Therefore A also measures B.

Q. E. D.

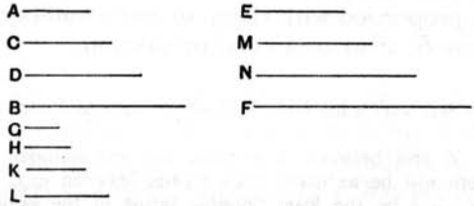
An obvious proof by *reductio ad absurdum* from VIII. 6.

PROPOSITION 8.

If between two numbers there fall numbers in continued proportion with them, then, however many numbers fall between them in continued proportion, so many will also fall in continued proportion between the numbers which have the same ratio with the original numbers.

Let the numbers C, D fall between the two numbers A, B in continued proportion with them, and let E be made in the same ratio to F as A is to B;

I say that, as many numbers as have fallen between A, B in continued proportion, so many will also fall between E, F in continued proportion.



For, as many as A, B, C, D are in multitude, let so many numbers G, H, K, L, the least of those which have the same ratio with A, C, D, B, be taken; [VII. 33]

therefore the extremes of them G, L are prime to one another. [VIII. 3]

Now, since A, C, D, B are in the same ratio with G, H, K, L,

and the multitude of the numbers A, C, D, B is equal to the multitude of the numbers G, H, K, L,

therefore, *ex aequali*, as A is to B, so is G to L. [VII. 14]

But, as A is to B, so is E to F;

therefore also, as G is to L, so is E to F.

But G, L are prime,

primes are also least, [VII. 21]

and the least numbers measure those which have the same ratio the same number of times, the greater the greater and the less the less, that is, the antecedent the antecedent and the consequent the consequent. [VII. 20]

Therefore G measures E the same number of times as L measures F.

Next, as many times as G measures E, so many times let H, K also measure M, N respectively;
therefore G, H, K, L measure E, M, N, F the same number of times.

Therefore G, H, K, L are in the same ratio with E, M, N, F. [VII. Def. 20]

But G, H, K, L are in the same ratio with A, C, D, B;
therefore A, C, D, B are also in the same ratio with E, M, N, F.

But A, C, D, B are in continued proportion;
therefore E, M, N, F are also in continued proportion.

Therefore, as many numbers as have fallen between A, B in continued proportion with them, so many numbers have also fallen between E, F in continued proportion.

Q. E. D.

1. **fall.** The Greek word is ἐμπίπτειν, "fall *in*" = "can be interpolated."

If $a : b = e : f$, and between a, b there are any number of geometric means c, d, there will be as many such means between e, f.

Let $\alpha, \beta, \gamma, \ldots, \delta$ be the least possible terms in the same ratio as $a, c, d, \ldots b$.

Then α, δ are prime to one another, [VIII. 3]

and, *ex aequali*,
$$\alpha : \delta = a : b$$
$$= e : f.$$

Therefore $e = m\alpha$, $f = m\delta$, where m is some integer. [VII. 20]

Take the numbers $m\alpha,\ m\beta,\ m\gamma,\ \ldots\ m\delta$.

This is a series in the given ratio, and we have the same number of geometric means between $m\alpha$, $m\delta$, or e, f, that there are between a, b.

Proposition 9.

If two numbers be prime to one another, and numbers fall between them in continued proportion, then, however many numbers fall between them in continued proportion, so many will also fall between each of them and an unit in continued proportion.

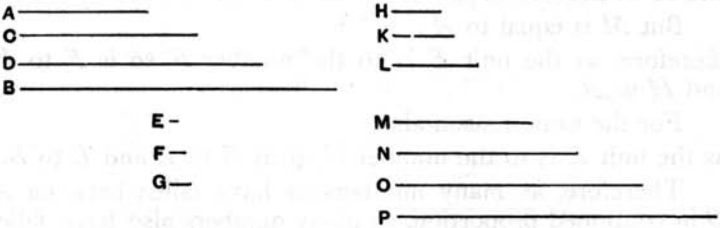
Let A, B be two numbers prime to one another, and let C, D fall between them in continued proportion,
and let the unit E be set out;
I say that, as many numbers as fall between A, B in con-

tinued proportion, so many will also fall between either of the numbers A, B and the unit in continued proportion.

For let two numbers F, G, the least that are in the ratio of A, C, D, B, be taken,

three numbers H, K, L with the same property,

and others more by one continually, until their multitude is equal to the multitude of A, C, D, B. [VIII. 2]



Let them be taken, and let them be M, N, O, P.

It is now manifest that F by multiplying itself has made H and by multiplying H has made M, while G by multiplying itself has made L and by multiplying L has made P. [VIII. 2, Por.]

And, since M, N, O, P are the least of those which have the same ratio with F, G,

and A, C, D, B are also the least of those which have the same ratio with F, G, [VIII. 1]

while the multitude of the numbers M, N, O, P is equal to the multitude of the numbers A, C, D, B,

therefore M, N, O, P are equal to A, C, D, B respectively;

therefore M is equal to A, and P to B.

Now, since F by multiplying itself has made H,

therefore F measures H according to the units in F.

But the unit E also measures F according to the units in it;

therefore the unit E measures the number F the same number of times as F measures H.

Therefore, as the unit E is to the number F, so is F to H. [VII. Def. 20]

Again, since F by multiplying H has made M,

therefore H measures M according to the units in F.

But the unit E also measures the number F according to the units in it;
therefore the unit E measures the number F the same number of times as H measures M.

Therefore, as the unit E is to the number F, so is H to M.

But it was also proved that, as the unit E is to the number F, so is F to H;
therefore also, as the unit E is to the number F, so is F to H, and H to M.

But M is equal to A;
therefore, as the unit E is to the number F, so is F to H, and H to A.

For the same reason also,
as the unit E is to the number G, so is G to L and L to B.

Therefore, as many numbers as have fallen between A, B in continued proportion, so many numbers also have fallen between each of the numbers A, B and the unit E in continued proportion.

Q. E. D.

Suppose there are n geometric means between a, b, two numbers prime to one another; there are the same number (n) of geometric means between 1 and a and between 1 and b.

If c, d... are the n means between a, b,

$$a, \quad c, \quad d \,\ldots\, b$$

are the least numbers in that ratio, since a, b are prime to one another. [VIII. 1]

The terms are therefore respectively identical with

$$\alpha^{n+1}, \quad \alpha^n\beta, \quad \alpha^{n-1}\beta^2 \ldots \alpha\beta^n, \quad \beta^{n+1},$$

where α, β is the common ratio in its lowest terms. [VIII. 2, Por.]

Thus $\qquad a = \alpha^{n+1}, \quad b = \beta^{n+1}.$

Now $\qquad 1 : \alpha = \alpha : \alpha^2 = \alpha^2 : \alpha^3 \ldots = \alpha^n : \alpha^{n+1},$

and $\qquad 1 : \beta = \beta : \beta^2 = \beta^2 : \beta^3 \ldots = \beta^n : \beta^{n+1};$

whence there are n geometric means between 1, a, and between 1, b.

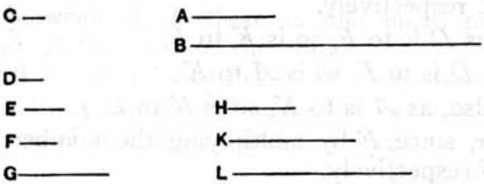
Proposition 10.

If numbers fall between each of two numbers and an unit in continued proportion, however many numbers fall between each of them and an unit in continued proportion, so many also will fall between the numbers themselves in continued proportion.

For let the numbers D, E and F, G respectively fall between the two numbers A, B and the unit C in continued proportion;

I say that, as many numbers as have fallen between each of the numbers A, B and the unit C in continued proportion, so many numbers will also fall between A, B in continued proportion.

For let D by multiplying F make H, and let the numbers D, F by multiplying H make K, L respectively.



Now, since, as the unit C is to the number D, so is D to E, therefore the unit C measures the number D the same number of times as D measures E. [VII. Def. 20]

But the unit C measures the number D according to the units in D;

therefore the number D also measures E according to the units in D;

therefore D by multiplying itself has made E.

Again, since, as C is to the number D, so is E to A, therefore the unit C measures the number D the same number of times as E measures A.

But the unit C measures the number D according to the units in D;

therefore E also measures A according to the units in D;

therefore D by multiplying E has made A.

For the same reason also

F by multiplying itself has made G, and by multiplying G has made B.

And, since D by multiplying itself has made E and by multiplying F has made H,

therefore, as D is to F, so is E to H. [VII. 17]

For the same reason also,

as D is to F, so is H to G. [VII. 18]

Therefore also, as E is to H, so is H to G.

Again, since D by multiplying the numbers E, H has made A, K respectively,

therefore, as E is to H, so is A to K. [VII. 17]

But, as E is to H, so is D to F;

therefore also, as D is to F, so is A to K.

Again, since the numbers D, F by multiplying H have made K, L respectively,

therefore, as D is to F, so is K to L. [VII. 18]

But, as D is to F, so is A to K;

therefore also, as A is to K, so is K to L.

Further, since F by multiplying the numbers H, G has made L, B respectively,

therefore, as H is to G, so is L to B. [VII. 17]

But, as H is to G, so is D to F;

therefore also, as D is to F, so is L to B.

But it was also proved that,

as D is to F, so is A to K and K to L;

therefore also, as A is to K, so is K to L and L to B.

Therefore A, K, L, B are in continued proportion.

Therefore, as many numbers as fall between each of the numbers A, B and the unit C in continued proportion, so many also will fall between A, B in continued proportion.

Q. E. D.

If there be n geometric means between 1 and a, and also between 1 and b, there will be n geometric means between a and b.

The proposition is the converse of the preceding.

The n means with the extremes form two geometric series of the form

$$1, \quad \alpha, \quad \alpha^2 \ldots \alpha^n, \quad \alpha^{n+1},$$
$$1, \quad \beta, \quad \beta^2 \ldots \beta^n, \quad \beta^{n+1},$$

where $$\alpha^{n+1} = a, \quad \beta^{n+1} = b.$$

By multiplying the last term in the first line by the first in the second, the last but one in the first line by the second in the second, and so on, we get the series

$$\alpha^{n+1}, \quad \alpha^n\beta, \quad \alpha^{n-1}\beta^2 \ldots \alpha^2\beta^{n-1}, \quad \alpha\beta^n, \quad \beta^{n+1}$$

and we have the n means between a and b.

It will be observed that, when Euclid says "*For the same reason* also, as D is to F, so is H to G," the reference is really to VII. 18 instead of VII. 17.

He infers namely that $D \times F : F \times F = D : F$. But since, by VII. 16, the order of multiplication is indifferent, he is practically justified in saying "for the same reason." The same thing occurs in later propositions.

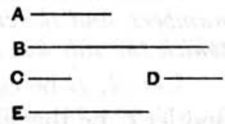
PROPOSITION 11.

Between two square numbers there is one mean proportional number, and the square has to the square the ratio duplicate of that which the side has to the side.

Let A, B be square numbers,

and let C be the side of A, and D of B;

I say that between A, B there is one mean proportional number, and A has to B the ratio duplicate of that which C has to D.



For let C by multiplying D make E.

Now, since A is a square and C is its side,

therefore C by multiplying itself has made A.

For the same reason also

D by multiplying itself has made B.

Since then C by multiplying the numbers C, D has made A, E respectively,

therefore, as C is to D, so is A to E. [VII. 17]

For the same reason also,

as C is to D, so is E to B. [VII. 18]

Therefore also, as A is to E, so is E to B.

Therefore between A, B there is one mean proportional number.

I say next that A also has to B the ratio duplicate of that which C has to D.

For, since A, E, B are three numbers in proportion, therefore A has to B the ratio duplicate of that which A has to E. [V. Def. 9]

But, as A is to E, so is C to D.

Therefore A has to B the ratio duplicate of that which the side C has to D. Q. E. D.

According to Nicomachus the theorems in this proposition and the next, that two squares have *one* geometric mean, and two cubes *two* geometric means, between them are Platonic. Cf. *Timaeus*, 32 A sqq. and the note thereon, p. 294 above.

a^2, b^2 being two squares, it is only necessary to form the product ab and to prove that

$$a^2,\ ab,\ b^2$$

are in geometrical progression. Euclid proves that

$$a^2 : ab = ab : b^2$$

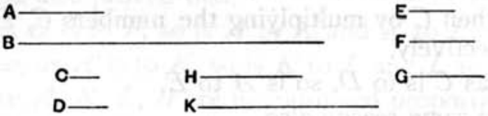
by means of VII. 17, 18, as usual.

In assuming that, since a^2 is to b^2 in the duplicate ratio of a^2 to ab, a^2 is to b^2 in the duplicate ratio of a to b, Euclid assumes that ratios which are the duplicates of equal ratios are equal. This, an obvious inference from V. 22, can be inferred just as easily for numbers from VII. 14.

PROPOSITION 12.

Between two cube numbers there are two mean proportional numbers, and the cube has to the cube the ratio triplicate of that which the side has to the side.

Let A, B be cube numbers,
and let C be the side of A, and D of B;
I say that between A, B there are two mean proportional numbers, and A has to B the ratio triplicate of that which C has to D.



For let C by multiplying itself make E, and by multiplying D let it make F;
let D by multiplying itself make G,
and let the numbers C, D by multiplying F make H, K respectively.

Now, since A is a cube, and C its side,
and C by multiplying itself has made E,
therefore C by multiplying itself has made E and by multiplying E has made A.

For the same reason also
D by multiplying itself has made G and by multiplying G has made B.

And, since C by multiplying the numbers C, D has made E, F respectively,
therefore, as C is to D, so is E to F. [VII. 17]

For the same reason also,

as C is to D, so is F to G. [VII. 18]

Again, since C by multiplying the numbers E, F has made A, H respectively,

therefore, as E is to F, so is A to H. [VII. 17]

But, as E is to F, so is C to D.

Therefore also, as C is to D, so is A to H.

Again, since the numbers C, D by multiplying F have made H, K respectively,

therefore, as C is to D, so is H to K. [VII. 18]

Again, since D by multiplying each of the numbers F, G has made K, B respectively,

therefore, as F is to G, so is K to B. [VII. 17]

But, as F is to G, so is C to D;

therefore also, as C is to D, so is A to H, H to K, and K to B.

Therefore H, K are two mean proportionals between A, B.

I say next that A also has to B the ratio triplicate of that which C has to D.

For, since A, H, K, B are four numbers in proportion, therefore A has to B the ratio triplicate of that which A has to H. [v. Def. 10]

But, as A is to H, so is C to D;

therefore A also has to B the ratio triplicate of that which C has to D.

Q. E. D.

The cube numbers a^3, b^3 being given, Euclid forms the products a^2b, ab^2 and then proves, as usual, by means of VII. 17, 18 that

$$a^3, \quad a^2b, \quad ab^2, \quad b^3$$

are in continued proportion.

He assumes that, since a^3 has to b^3 the ratio triplicate of $a^3 : a^2b$, the ratio $a^3 : b^3$ is triplicate of the ratio $a : b$ which is equal to $a^3 : a^2b$. This is again an obvious inference from VII. 14.

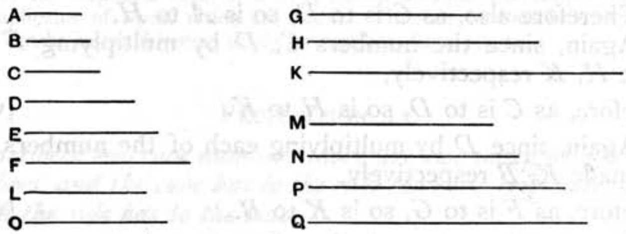
PROPOSITION 13.

If there be as many numbers as we please in continued proportion, and each by multiplying itself make some number, the products will be proportional; and, if the original numbers by multiplying the products make certain numbers, the latter will also be proportional.

Let there be as many numbers as we please, A, B, C, in continued proportion, so that, as A is to B, so is B to C;

let A, B, C by multiplying themselves make D, E, F, and by multiplying D, E, F let them make G, H, K;

I say that D, E, F and G, H, K are in continued proportion.



For let A by multiplying B make L,

and let the numbers A, B by multiplying L make M. N respectively.

And again let B by multiplying C make O,

and let the numbers B, C by multiplying O make P, Q respectively.

Then, in manner similar to the foregoing, we can prove that

D, L, E and G, M, N, H are continuously proportional in the ratio of A to B,

and further E, O, F and H, P, Q, K are continuously proportional in the ratio of B to C.

Now, as A is to B, so is B to C;

therefore D, L, E are also in the same ratio with E, O, F,

and further G, M, N, H in the same ratio with H, P, Q, K.

And the multitude of D, L, E is equal to the multitude of E, O, F, and that of G, M, N, H to that of H, P, Q, K;

therefore, *ex aequali*,

as D is to E, so is E to F,

and, as G is to H, so is H to K. [VII. 14]

If a, b, c ... be a series in geometrical progression, then

$$\left.\begin{array}{l} a^2,\ b^2,\ c^2 \ldots \\ a^3,\ b^3,\ c^3 \ldots \end{array}\right\} \text{ are also in geometrical progression.}$$

and

Heiberg brackets the words added to the enunciation which extend the theorem to any powers. The words are "and this always occurs with the extremes" (καὶ ἀεὶ περὶ τοὺς ἄκρους τοῦτο συμβαίνει). They seem to be rightly suspected on the same grounds as the same words added to the enunciation of VII. 27. There is no allusion to them in the proof, much less any proof of the extension.

Euclid forms, besides the squares and cubes of the given numbers, the products ab, a^2b, ab^2, bc, b^2c, bc^2. When he says that "we prove in manner similar to the foregoing," he indicates successive uses of VII. 17, 18 as in VIII. 12.

With our notation the proof is as easy to see for *any* powers as for squares and cubes.

To prove that a^n, b^n, c^n... are in geometrical progression.

Form all the means between a^n, b^n, and set out the series

$$a^n,\quad a^{n-1}b,\quad a^{n-2}b^2 \ldots ab^{n-1},\quad b^n.$$

The common ratio of one term to the next is $a : b$.

Next take the geometrical progression

$$b^n,\quad b^{n-1}c,\quad b^{n-2}c^2 \ldots bc^{n-1},\quad c^n,$$

the common ratio of which is $b : c$.

Proceed thus for all pairs of consecutive terms.

Now $$a : b = b : c = \ldots$$

Therefore any pair of succeeding terms in one series are in the same ratio as any pair of succeeding terms in any other of the series.

And the number of terms in each is the same, namely $(n+1)$.

Therefore, *ex aequali*,

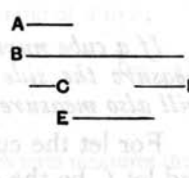
$$a^n : b^n = b^n : c^n = c^n : d^n = \ldots$$

PROPOSITION 14.

If a square measure a square, the side will also measure the side; and, if the side measure the side, the square will also measure the square.

Let A, B be square numbers, let C, D be their sides, and let A measure B;

I say that C also measures D.



For let C by multiplying D make E; therefore A, E, B are continuously proportional in the ratio of C to D. [VIII. 11]

And, since A, E, B are continuously proportional, and A measures B, therefore A also measures E. [VIII. 7]

And, as A is to E, so is C to D;
therefore also C measures D. [VII. Def. 20]

Again, let C measure D;
I say that A also measures B.

For, with the same construction, we can in a similar manner prove that A, E, B are continuously proportional in the ratio of C to D.

And since, as C is to D, so is A to E,
and C measures D,
therefore A also measures E. [VII. Def. 20]

And A, E, B are continuously proportional;
therefore A also measures B.

Therefore etc.

Q. E. D.

If a^2 measures b^2, a measures b; and, if a measures b, a^2 measures b^2.

(1) a^2, ab, b^2 are in continued proportion in the ratio of a to b.

Therefore, since a^2 measures b^2,
a^2 measures ab. [VIII. 7]
But $a^2 : ab = a : b$.
Therefore a measures b.

(2) Since a measures b, a^2 measures ab.
And a^2, ab, b^2 are continuously proportional.
Thus ab measures b^2.
And a^2 measures ab.
Therefore a^2 measures b^2.

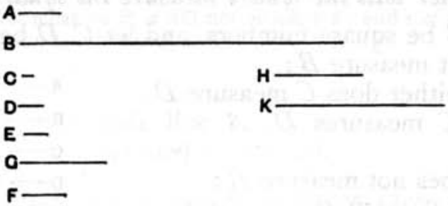
It will be seen that Euclid puts the last step shortly, saying that, since a^2 measures ab, and a^2, ab, b^2 are in continued proportion, a^2 measures b^2. The same thing happens in VIII. 15, where the series of terms is one more than here.

Proposition 15.

If a cube number measure a cube number, the side will also measure the side; and, if the side measure the side, the cube will also measure the cube.

For let the cube number A measure the cube B,
and let C be the side of A and D of B;
I say that C measures D.

For let C by multiplying itself make E,
and let D by multiplying itself make G;
further, let C by multiplying D make F,
and let C, D by multiplying F make H, K respectively.



Now it is manifest that E, F, G and A, H, K, B are continuously proportional in the ratio of C to D. [VIII. 11, 12]

And, since A, H, K, B are continuously proportional,
and A measures B,
therefore it also measures H. [VIII. 7]

And, as A is to H, so is C to D;
therefore C also measures D. [VII. Def. 20]

Next, let C measure D;
I say that A will also measure B.

For, with the same construction, we can prove in a similar manner that A, H, K, B are continuously proportional in the ratio of C to D.

And, since C measures D,
and, as C is to D, so is A to H,
therefore A also measures H, [VII. Def. 20]
so that A measures B also.

Q. E. D.

If a^3 measures b^3, a measures b; and *vice versa*. The proof is, *mutatis mutandis*, the same as for squares.

(1) a^3, a^2b, ab^2, b^3 are continuously proportional in the ratio of a to b;
and a^3 measures b^3.

Therefore a^3 measures a^2b; [VIII. 7]
and hence a measures b.

(2) Since a measures b, a^3 measures a^2b.

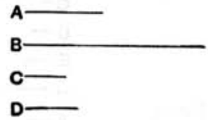
And, a^3, a^2b, ab^2, b^3 being continuously proportional, each term measures the succeeding term;
therefore a^3 measures b^3.

Proposition 16.

If a square number do not measure a square number, neither will the side measure the side; and, if the side do not measure the side, neither will the square measure the square.

Let A, B be square numbers, and let C, D be their sides; and let A not measure B;

I say that neither does C measure D.



For, if C measures D, A will also measure B. [VIII. 14]

But A does not measure B;

therefore neither will C measure D.

Again, let C not measure D;

I say that neither will A measure B.

For, if A measures B, C will also measure D. [VIII. 14]

But C does not measure D;

therefore neither will A measure B.

Q. E. D.

If a^2 does not measure b^2, a will not measure b; and, if a does not measure b, a^2 will not measure b^2.

The proof is a mere *reductio ad absurdum* using VIII. 14.

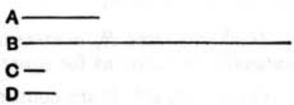
Proposition 17.

If a cube number do not measure a cube number, neither will the side measure the side; and, if the side do not measure the side, neither will the cube measure the cube.

For let the cube number A not measure the cube number B,

and let C be the side of A, and D of B;

I say that C will not measure D.



For if C measures D, A will also measure B. [VIII. 15]

But A does not measure B;

therefore neither does C measure D.

Again, let C not measure D;

I say that neither will A measure B.

For, if A measures B, C will also measure D. [VIII. 15]

But C does not measure D;

therefore neither will A measure B.

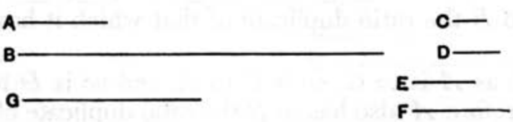
Q. E. D.

If a^3 does not measure b^3, a will not measure b; and *vice versa*. Proved by *reductio ad absurdum* employing VIII. 15.

PROPOSITION 18.

Between two similar plane numbers there is one mean proportional number; and the plane number has to the plane number the ratio duplicate of that which the corresponding side has to the corresponding side.

Let A, B be two similar plane numbers, and let the numbers C, D be the sides of A, and E, F of B.



Now, since similar plane numbers are those which have their sides proportional, [VII. Def. 21]

therefore, as C is to D, so is E to F.

I say then that between A, B there is one mean proportional number, and A has to B the ratio duplicate of that which C has to E, or D to F, that is, of that which the corresponding side has to the corresponding side.

Now since, as C is to D, so is E to F,

therefore, alternately, as C is to E, so is D to F. [VII. 13]

And, since A is plane, and C, D are its sides,

therefore D by multiplying C has made A.

For the same reason also

E by multiplying F has made B.

Now let D by multiplying E make G.

Then, since D by multiplying C has made A, and by multiplying E has made G,

therefore, as C is to E, so is A to G. [VII. 17]

But, as C is to E, so is D to F;
therefore also, as D is to F, so is A to G.

Again, since E by multiplying D has made G, and by multiplying F has made B,
therefore, as D is to F, so is G to B. [VII. 17]

But it was also proved that,

as D is to F, so is A to G;

therefore also, as A is to G, so is G to B.

Therefore A, G, B are in continued proportion.

Therefore between A, B there is one mean proportional number.

I say next that A also has to B the ratio duplicate of that which the corresponding side has to the corresponding side, that is, of that which C has to E or D to F.

For, since A, G, B are in continued proportion,
A has to B the ratio duplicate of that which it has to G. [V. Def. 9]

And, as A is to G, so is C to E, and so is D to F.

Therefore A also has to B the ratio duplicate of that which C has to E or D to F.

Q. E. D.

If ab, cd be "similar plane numbers," i.e. products of factors such that

$$a : b = c : d,$$

there is one mean proportional between ab and cd; and ab is to cd in the duplicate ratio of a to c or of b to d.

Form the product bc (or ad, which is equal to it, by VII. 19).

Then $$ab,\ \left.\begin{matrix} bc \\ = ad \end{matrix}\right\},\ cd$$

is a series of terms in geometrical progression.

For $a : b = c : d$.

Therefore $a : c = b : d$. [VII. 13]

Therefore $ab : bc = bc : cd$. [VII. 17 and 16]

Thus bc (or ad) is a geometric mean between ab, cd.

And ab is to cd in the duplicate ratio of ab to bc or of bc to cd, that is, of a to c or of b to d.

Proposition 19.

Between two similar solid numbers there fall two mean proportional numbers; and the solid number has to the similar solid number the ratio triplicate of that which the corresponding side has to the corresponding side.

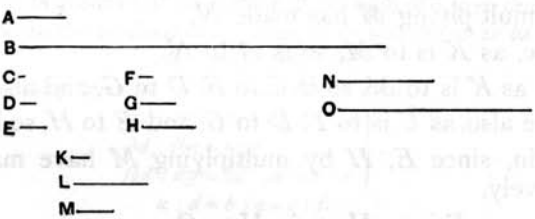
Let A, B be two similar solid numbers, and let C, D, E be the sides of A, and F, G, H of B.

Now, since similar solid numbers are those which have their sides proportional, [VII. Def. 21]

therefore, as C is to D, so is F to G,

and, as D is to E, so is G to H.

I say that between A, B there fall two mean proportional numbers, and A has to B the ratio triplicate of that which C has to F, D to G, and also E to H.



For let C by multiplying D make K, and let F by multiplying G make L.

Now, since C, D are in the same ratio with F, G, and K is the product of C, D, and L the product of F, G, K, L are similar plane numbers; [VII. Def. 21]

therefore between K, L there is one mean proportional number. [VIII. 18]

Let it be M.

Therefore M is the product of D, F, as was proved in the theorem preceding this. [VIII. 18]

Now, since D by multiplying C has made K, and by multiplying F has made M,

therefore, as C is to F, so is K to M. [VII. 17]

But, as K is to M, so is M to L.

Therefore K, M, L are continuously proportional in the ratio of C to F.

And since, as C is to D, so is F to G,
alternately therefore, as C is to F, so is D to G. [VII. 13]

For the same reason also,

as D is to G, so is E to H.

Therefore K, M, L are continuously proportional in the ratio of C to F, in the ratio of D to G, and also in the ratio of E to H.

Next, let E, H by multiplying M make N, O respectively.

Now, since A is a solid number, and C, D, E are its sides, therefore E by multiplying the product of C, D has made A.

But the product of C, D is K;
therefore E by multiplying K has made A.

For the same reason also

H by multiplying L has made B.

Now, since E by multiplying K has made A, and further also by multiplying M has made N,
therefore, as K is to M, so is A to N. [VII. 17]

But, as K is to M, so is C to F, D to G, and also E to H;
therefore also, as C is to F, D to G, and E to H, so is A to N.

Again, since E, H by multiplying M have made N, O respectively,
therefore, as E is to H, so is N to O. [VII. 18]

But, as E is to H, so is C to F and D to G;
therefore also, as C is to F, D to G, and E to H, so is A to N and N to O.

Again, since H by multiplying M has made O, and further also by multiplying L has made B,
therefore, as M is to L, so is O to B. [VII. 17]

But, as M is to L, so is C to F, D to G, and E to H.

Therefore also, as C is to F, D to G, and E to H, so not only is O to B, but also A to N and N to O.

Therefore A, N, O, B are continuously proportional in the aforesaid ratios of the sides.

I say that A also has to B the ratio triplicate of that which the corresponding side has to the corresponding side, that is, of the ratio which the number C has to F, or D to G, and also E to H.

For, since A, N, O, B are four numbers in continued proportion,

therefore A has to B the ratio triplicate of that which A has to N. [v. Def. 10]

But, as A is to N, so it was proved that C is to F, D to G, and also E to H.

Therefore A also has to B the ratio triplicate of that which the corresponding side has to the corresponding side, that is, of the ratio which the number C has to F, D to G, and also E to H. Q. E. D.

In other words, if $a : b : c = d : e : f$, then there are two geometric means between abc, def; and abc is to def in the triplicate ratio of a to d, or b to e, or c to f.

Euclid first takes the plane numbers ab, de (leaving out c, f) and forms the product bd. Thus, as in VIII. 18,

$$\left.\begin{matrix} ab, & bd \\ & = ea \end{matrix}\right\}, \quad de$$

are three terms in geometrical progression in the ratio of a to d, or of b to e.

He next forms the products of c, f respectively into the mean bd.

Then $$abc, \quad cbd, \quad fbd, \quad def$$

are in geometrical progression in the ratio of a to d etc.

For
$$\left.\begin{aligned} abc : cbd &= ab : bd = a : d \\ bd : fbd &= c : f \\ fbd : def &= bd : de = b : e \end{aligned}\right\}. \qquad \text{[VII. 17]}$$

And $$a : d = b : e = c : f.$$

The ratio of abc to def is the ratio triplicate of that of abc to cbd, i.e. of that of a to d etc.

Proposition 20.

If one mean proportional number fall between two numbers, the numbers will be similar plane numbers.

For let one mean proportional number C fall between the two numbers A, B;

I say that A, B are similar plane numbers.

Let D, E, the least numbers of those which have the same ratio with A, C, be taken; [VII. 33]

therefore D measures A the same number of times that E measures C. [VII. 20]

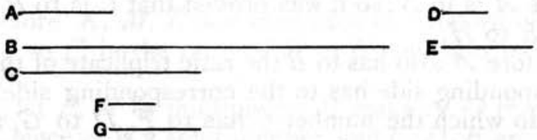
Now, as many times as D measures A, so many units let there be in F;

therefore F by multiplying D has made A,

so that A is plane, and D, F are its sides.

Again, since D, E are the least of the numbers which have the same ratio with C, B,

therefore D measures C the same number of times that E measures B. [VII. 20]



As many times, then, as E measures B, so many units let there be in G;

therefore E measures B according to the units in G;

therefore G by multiplying E has made B.

Therefore B is plane, and E, G are its sides.

Therefore A, B are plane numbers.

I say next that they are also similar.

For, † since F by multiplying D has made A, and by multiplying E has made C,

therefore, as D is to E, so is A to C, that is, C to B. [VII. 17]

Again, † since E by multiplying F, G has made C, B respectively,

therefore, as F is to G, so is C to B. [VII. 17]

But, as C is to B, so is D to E;

therefore also, as D is to E, so is F to G.

And alternately, as D is to F, so is E to G. [VII. 13]

Therefore A, B are similar plane numbers; for their sides are proportional. Q. E. D.

25. **For, since F......27. C to B.** The text has clearly suffered corruption here. It is not necessary to *infer* from other facts that, as D is to E, so is A to C; for this is part of the hypotheses (ll. 6, 7). Again, there is no explanation of the statement (l. 25) that F by multiplying E has made C. It is the statement and explanation of this latter fact which are alone wanted; after which the proof proceeds as in l. 28. We might therefore substitute for ll. 25—28 the following.

"For, since E measures C the same number of times that D measures A [l. 8], that is, according to the units in F [l. 10], therefore F by multiplying E has made C.
And, since E by multiplying F, G," etc. etc.

This proposition is the converse of VIII. 18. If a, c, b are in geometrical progression, a, b are "similar plane numbers."

Let $\alpha : \beta$ be the ratio $a : c$ (and therefore also the ratio $c : b$) in its lowest terms.

Then [VII. 20]

$$a = m\alpha, \quad c = m\beta, \quad \text{where } m \text{ is some integer,}$$
$$c = n\alpha, \quad b = n\beta, \quad \text{where } n \text{ is some integer.}$$

Thus a, b are both products of two factors, i.e. plane.

Again, $a : \beta = a : c = c : b$

$= m : n.$ [VII. 18]

Therefore, alternately, $a : m = \beta : n$, [VII. 13]

and hence $m\alpha$, $n\beta$ are *similar* plane numbers.

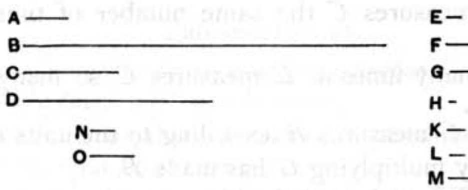
[Our notation makes the second part still more obvious, for $c = m\beta = n\alpha$.]

PROPOSITION 21.

If two mean proportional numbers fall between two numbers, the numbers are similar solid numbers.

For let two mean proportional numbers C, D fall between the two numbers A, B;

I say that A, B are similar solid numbers.



For let three numbers E, F, G, the least of those which have the same ratio with A, C, D, be taken; [VII. 33 or VIII. 2]

therefore the extremes of them E, G are prime to one another. [VIII. 3]

Now, since one mean proportional number F has fallen between E, G,

therefore E, G are similar plane numbers. [VIII. 20]

Let, then, H, K be the sides of E, and L, M of G.

Therefore it is manifest from the theorem before this that E, F, G are continuously proportional in the ratio of H to L and that of K to M.

Now, since E, F, G are the least of the numbers which have the same ratio with A, C, D,

and the multitude of the numbers E, F, G is equal to the multitude of the numbers A, C, D,

therefore, *ex aequali*, as E is to G, so is A to D. [VII. 14]

But E, G are prime,

primes are also least, [VII. 21]

and the least measure those which have the same ratio with

them the same number of times, the greater the greater and the less the less, that is, the antecedent the antecedent and the consequent the consequent; [VII. 20]
therefore E measures A the same number of times that G measures D.

Now, as many times as E measures A, so many units let there be in N.

Therefore N by multiplying E has made A.

But E is the product of H, K;
therefore N by multiplying the product of H, K has made A.

Therefore A is solid, and H, K, N are its sides.

Again, since E, F, G are the least of the numbers which have the same ratio as C, D, B,
therefore E measures C the same number of times that G measures B.

Now, as many times as E measures C, so many units let there be in O.

Therefore G measures B according to the units in O;
therefore O by multiplying G has made B.

But G is the product of L, M;
therefore O by multiplying the product of L, M has made B.

Therefore B is solid, and L, M, O are its sides;
therefore A, B are solid.

I say that they are also similar.

For since N, O by multiplying E have made A, C,
therefore, as N is to O, so is A to C, that is, E to F. [VII. 18]

But, as E is to F, so is H to L and K to M;
therefore also, as H is to L, so is K to M and N to O.

And H, K, N are the sides of A, and O, L, M the sides of B.

Therefore A, B are similar solid numbers. Q. E. D.

The converse of VIII. 19. If a, c, d, b are in geometrical progression, a, b are "similar solid numbers."

Let α, β, γ be the least numbers in the ratio of a, c, d (and therefore also of c, d, b). [VII. 33 or VIII. 2]

Therefore α, γ are prime to one another. [VIII. 3]

They are also "similar plane numbers." [VIII. 20]

Let $$\alpha = mn, \quad \gamma = pq,$$
where $$m : n = p : q.$$

Then, by the proof of VIII. 20,

$$\alpha : \beta = m : p = n : q.$$

Now, *ex aequali*, $a : d = \alpha : \gamma$, [VII. 14]

and, since α, γ are prime to one another,

$$a = r\alpha, \quad d = r\gamma, \quad \text{where } r \text{ is an integer.}$$

But $\alpha = mn$:

therefore $a = rmn$, and therefore a is "solid."

Again, *ex aequali*, $c : b = \alpha : \gamma$,

and therefore $c = s\alpha$, $b = s\gamma$, where s is an integer.

Thus $b = spq$, and b is therefore "solid."

Now $\alpha : \beta = a : c = r\alpha : s\alpha$

$= r : s$. [VII. 18]

And, from above, $\alpha : \beta = m : p = n : q$.

Therefore $r : s = m : p = n : q$,

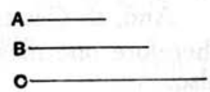
and hence a, b are *similar* solid numbers.

PROPOSITION 22.

If three numbers be in continued proportion, and the first be square, the third will also be square.

Let A, B, C be three numbers in continued proportion, and let A the first be square;

I say that C the third is also square.



For, since between A, C there is one mean proportional number, B,

therefore A, C are similar plane numbers. [VIII. 20]

But A is square;

therefore C is also square. Q. E. D.

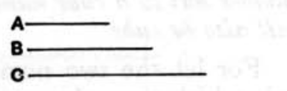
A mere application of VIII. 20 to the particular case where one of the "similar plane numbers" is square.

PROPOSITION 23.

If four numbers be in continued proportion, and the first be cube, the fourth will also be cube.

Let A, B, C, D be four numbers in continued proportion, and let A be cube;

I say that D is also cube.



For, since between A, D there are two mean proportional numbers B, C,

therefore A, D are similar solid numbers. [VIII. 21]

But A is cube;
therefore D is also cube.

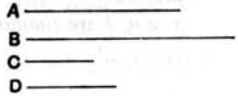
Q. E. D.

A mere application of VIII. 21 to the case where one of the "similar solid numbers" is a cube.

PROPOSITION 24.

If two numbers have to one another the ratio which a square number has to a square number, and the first be square, the second will also be square.

For let the two numbers A, B have to one another the ratio which the square number C has to the square number D, and let A be square;



I say that B is also square.

For, since C, D are square,
C, D are similar plane numbers.

Therefore one mean proportional number falls between C, D. [VIII. 18]

And, as C is to D, so is A to B;
therefore one mean proportional number falls between A, B also. [VIII. 8]

And A is square;
therefore B is also square. [VIII. 22]

Q. E. D.

If $a : b = c^2 : d^2$, and a is a square, then b is also a square.
For c^2, d^2 have one mean proportional cd. [VIII. 18]
Therefore a, b, which are in the same ratio, have one mean proportional. [VIII. 8]
And, since a is square, b must also be a square. [VIII. 22]

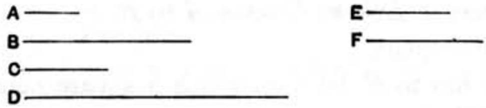
PROPOSITION 25.

If two numbers have to one another the ratio which a cube number has to a cube number, and the first be cube, the second will also be cube.

For let the two numbers A, B have to one another the ratio which the cube number C has to the cube number D, and let A be cube;
I say that B is also cube.

For, since C, D are cube,
C, D are similar solid numbers.

Therefore two mean proportional numbers fall between C, D. [VIII. 19]



And, as many numbers as fall between C, D in continued proportion, so many will also fall between those which have the same ratio with them; [VIII. 8]
so that two mean proportional numbers fall between A, B also.

Let E, F so fall.

Since, then, the four numbers A, E, F, B are in continued proportion,
and A is cube,
therefore B is also cube. [VIII. 23]

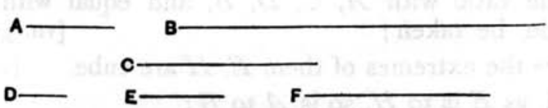
Q. E. D.

If $a : b = c^3 : d^3$, and a is a cube, then b is also a cube.
For c^3, d^3 have two mean proportionals. [VIII. 19]
Therefore a, b also have two mean proportionals. [VIII. 8]
And a is a cube:
therefore b is a cube. [VIII. 23]

PROPOSITION 26.

Similar plane numbers have to one another the ratio which a square number has to a square number.

Let A, B be similar plane numbers;
I say that A has to B the ratio which a square number has to a square number.



For, since A, B are similar plane numbers,
therefore one mean proportional number falls between A, B. [VIII. 18]

Let it so fall, and let it be C;
and let D, E, F, the least numbers of those which have the same ratio with A, C, B, be taken; [VII. 33 or VIII. 2]
therefore the extremes of them D, F are square. [VIII. 2, Por.]

And since, as D is to F, so is A to B,
and D, F are square,
therefore A has to B the ratio which a square number has to a square number.

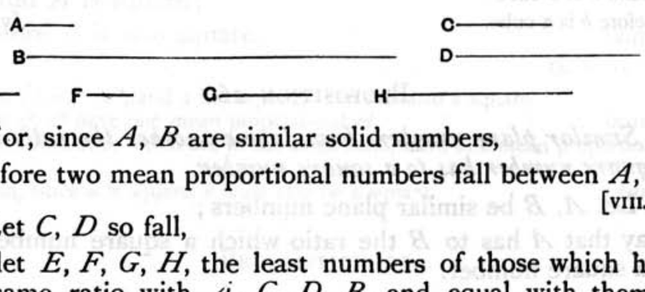
Q. E. D.

If a, b are similar "plane numbers," let c be the mean proportional between them. [VIII. 18]
Take α, β, γ the smallest numbers in the ratio of a, c, b. [VII. 33 or VIII. 2]
Then α, γ are squares. [VIII. 2, Por.]
Therefore a, b are in the ratio of a square to a square.

Proposition 27.

Similar solid numbers have to one another the ratio which a cube number has to a cube number.

Let A, B be similar solid numbers;
I say that A has to B the ratio which a cube number has to a cube number.



For, since A, B are similar solid numbers,
therefore two mean proportional numbers fall between A, B. [VIII. 19]

Let C, D so fall,
and let E, F, G, H, the least numbers of those which have the same ratio with A, C, D, B, and equal with them in multitude, be taken; [VII. 33 or VIII. 2]
therefore the extremes of them E, H are cube. [VIII. 2, Por.]

And, as E is to H, so is A to B;
therefore A also has to B the ratio which a cube number has to a cube number.

Q. E. D.

The same thing as VIII. 26 with cubes. It is proved in the same way except that VIII. 19 is used instead of VIII. 18.

The last note of an-Nairīzī in which the name of Heron is mentioned is on this proposition. Heron is there stated (p. 194—5, ed. Curtze) to have added the two propositions that,

1. *If two numbers have to one another the ratio of a square to a square, the numbers are similar plane numbers;*

2. *If two numbers have to one another the ratio of a cube to a cube, the numbers are similar solid numbers.*

The propositions are of course the converses of VIII. 26, 27 respectively. They are easily proved.

(1) If $$a : b = c^2 : d^2,$$
then, since there is one mean proportional (cd) between c^2, d^2, [VIII. 11 or 18]
there is also one mean proportional between a, b. [VIII. 8]

Therefore a, b are similar plane numbers. [VIII. 20]

(2) is similarly proved by the use of VIII. 12 or 19, VIII. 8, VIII. 21.

The insertion by Heron of the first of the two propositions, the converse of VIII. 26, is perhaps an argument in favour of the correctness of the text of IX. 10, though (as remarked in the note on that proposition) it does not give the easiest proof Cf. Heron's extension of VII. 3 tacitly assumed by Euclid in VII. 33.

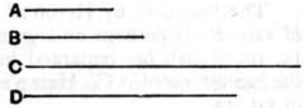
BOOK IX.

Proposition 1.

If two similar plane numbers by multiplying one another make some number, the product will be square.

Let A, B be two similar plane numbers, and let A by multiplying B make C;

I say that C is square.



For let A by multiplying itself make D.

Therefore D is square.

Since then A by multiplying itself has made D, and by multiplying B has made C,

therefore, as A is to B, so is D to C. [VII. 17]

And, since A, B are similar plane numbers,

therefore one mean proportional number falls between A, B. [VIII. 18]

But, if numbers fall between two numbers in continued proportion, as many as fall between them, so many also fall between those which have the same ratio; [VIII. 8]

so that one mean proportional number falls between D, C also.

And D is square;

therefore C is also square. [VIII. 22]

Q. E. D.

The product of two similar plane numbers is a square.

Let a, b be two similar plane numbers.

Now $a : b = a^2 : ab$. [VII. 17]

And between a, b there is one mean proportional. [VIII. 18]

Therefore between $a^2 : ab$ there is one mean proportional. [VIII. 8]

And a^2 is square;

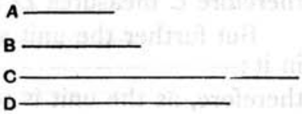
therefore ab is square. [VIII. 22]

Proposition 2.

If two numbers by multiplying one another make a square number, they are similar plane numbers.

Let A, B be two numbers, and let A by multiplying B make the square number C;

I say that A, B are similar plane numbers.



For let A by multiplying itself make D;

therefore D is square.

Now, since A by multiplying itself has made D, and by multiplying B has made C,

therefore, as A is to B, so is D to C. [VII. 17]

And, since D is square, and C is so also,

therefore D, C are similar plane numbers.

Therefore one mean proportional number falls between D, C. [VIII. 18]

And, as D is to C, so is A to B;

therefore one mean proportional number falls between A, B also. [VIII. 8]

But, if one mean proportional number fall between two numbers, they are similar plane numbers; [VIII. 20]

therefore A, B are similar plane numbers.

Q. E. D.

If ab is a square number, a, b are similar plane numbers. (The converse of IX. 1.)

For $$a : b = a^2 : ab.$$ [VII. 17]

And a^2, ab being square numbers, and therefore similar plane numbers, they have one mean proportional. [VIII. 18]

Therefore a, b also have one mean proportional. [VIII. 8]

whence a, b are similar plane numbers. [VIII. 20]

Proposition 3.

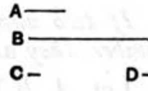
If a cube number by multiplying itself make some number, the product will be cube.

For let the cube number A by multiplying itself make B;

I say that B is cube.

For let C, the side of A, be taken, and let C by multiplying itself make D.

It is then manifest that C by multiplying D has made A.



Now, since C by multiplying itself has made D,
therefore C measures D according to the units in itself.

But further the unit also measures C according to the units in it;
therefore, as the unit is to C, so is C to D. [VII. Def. 20]

Again, since C by multiplying D has made A,
therefore D measures A according to the units in C.

But the unit also measures C according to the units in it;
therefore, as the unit is to C, so is D to A.

But, as the unit is to C, so is C to D;
therefore also, as the unit is to C, so is C to D, and D to A.

Therefore between the unit and the number A two mean proportional numbers C, D have fallen in continued proportion.

Again, since A by multiplying itself has made B,
therefore A measures B according to the units in itself.

But the unit also measures A according to the units in it;
therefore, as the unit is to A, so is A to B. [VII. Def. 20]

But between the unit and A two mean proportional numbers have fallen;
therefore two mean proportional numbers will also fall between A, B. [VIII. 8]

But, if two mean proportional numbers fall between two numbers, and the first be cube, the second will also be cube. [VIII. 23]

And A is cube;
therefore B is also cube. Q. E. D.

The product of a^3 into itself, or $a^3 . a^3$, is a cube.

For $1 : a = a : a^2 = a^2 : a^3$.

Therefore between 1 and a^3 there are two mean proportionals.

Also $1 : a^3 = a^3 : a^3 . a^3$.

Therefore two mean proportionals fall between a^3 and $a^3 . a^3$. [VIII. 8]
(It is true that VIII. 8 is only enunciated of two pairs of numbers, but the proof is equally valid if one number of one pair is unity.)

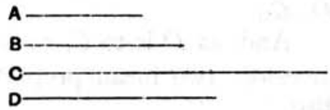
And a^3 is a cube number:
therefore $a^3 . a^3$ is also cube. [VIII. 23]

Proposition 4.

If a cube number by multiplying a cube number make some number, the product will be cube.

For let the cube number A by multiplying the cube number B make C;

I say that C is cube.



For let A by multiplying itself make D;

therefore D is cube. [IX. 3]

And, since A by multiplying itself has made D, and by multiplying B has made C therefore, as A is to B, so is D to C. [VII. 17]

And, since A, B are cube numbers,

A, B are similar solid numbers.

Therefore two mean proportional numbers fall between A, B; [VIII. 19]

so that two mean proportional numbers will fall between D, C also. [VIII. 8]

And D is cube;

therefore C is also cube [VIII. 23]

Q. E. D.

The product of two cubes, say $a^3 . b^3$, is a cube.

For $a^3 : b^3 = a^3 . a^3 : a^3 . b^3$. [VII. 17]

And two mean proportionals fall between a^3, b^3 which are similar solid numbers. [VIII. 19]

Therefore two mean proportionals fall between $a^3 . a^3$, $a^3 . b^3$ [VIII. 8]

But $a^3 . a^3$ is a cube: [IX. 3]

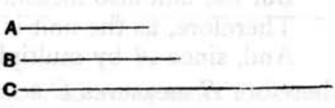
therefore $a^3 . b^3$ is a cube. [VIII. 23]

Proposition 5.

If a cube number by multiplying any number make a cube number, the multiplied number will also be cube.

For let the cube number A by multiplying any number B make the cube number C;

I say that B is cube.



For let A by multiplying itself make D;

therefore D is cube. [IX. 3]

Now, since A by multiplying itself has made D, and by multiplying B has made C,

therefore, as A is to B, so is D to C. [VII. 17]

And since D, C are cube,

they are similar solid numbers.

Therefore two mean proportional numbers fall between D, C. [VIII. 19]

And, as D is to C, so is A to B;

therefore two mean proportional numbers fall between A, B also. [VIII. 8]

And A is cube;

therefore B is also cube. [VIII. 23]

If the product a^3b is a cube number, b is cube.

By IX. 3, the product $a^3 . a^3$ is a cube.

And $a^3 . a^3 : a^3b = a^3 : b$. [VII. 17]

The first two terms are cubes, and therefore "similar solids"; therefore there are two mean proportionals between them. [VIII. 19]

Therefore there are two mean proportionals between a^3, b. [VIII. 8]

And a^3 is a cube:

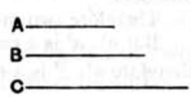
therefore b is a cube number. [VIII. 23]

Proposition 6.

If a number by multiplying itself make a cube number, it will itself also be cube.

For let the number A by multiplying itself make the cube number B;

I say that A is also cube.



For let A by multiplying B make C.

Since, then, A by multiplying itself has made B, and by multiplying B has made C,

therefore C is cube.

And, since A by multiplying itself has made B,

therefore A measures B according to the units in itself.

But the unit also measures A according to the units in it.

Therefore, as the unit is to A, so is A to B. [VII. Def. 20]

And, since A by multiplying B has made C,

therefore B measures C according to the units in A.

But the unit also measures A according to the units in it.

Therefore, as the unit is to A, so is B to C. [VII. Def. 20]

But, as the unit is to A, so is A to B;
therefore also, as A is to B, so is B to C.

And, since B, C are cube,
they are similar solid numbers.

Therefore there are two mean proportional numbers between B, C. [VIII. 19]

And, as B is to C, so is A to B.

Therefore there are two mean proportional numbers between A, B also. [VIII. 8]

And B is cube;
therefore A is also cube. [cf. VIII. 23]

Q. E. D.

If a^2 is a cube number, a is also a cube.

For $$1 : a = a : a^2 = a^2 : a^3.$$

Now a^2, a^3 are both cubes, and therefore "similar solids"; therefore there are two mean proportionals between them. [VIII. 19]

Therefore there are two mean proportionals between a, a^2. [VIII. 8]

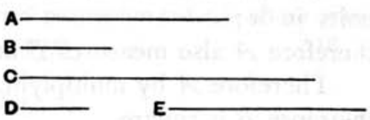
And a^2 is a cube:
therefore a is also a cube number. [VIII. 23]

It will be noticed that the last step is not an exact quotation of the result of VIII. 23, because it is there the *first* of four terms which is known to be a cube, and the *last* which is proved to be a cube; here the case is reversed. But there is no difficulty. Without inverting the proportions, we have only to refer to VIII. 21 which proves that a, a^2, having two mean proportionals between them, are two similar solid numbers; whence, since a^2 is a cube, a is also a cube.

PROPOSITION 7.

If a composite number by multiplying any number make some number, the product will be solid.

For let the composite number A by multiplying any number B make C;
I say that C is solid.



For, since A is composite, it will be measured by some number. [VII. Def. 13]

Let it be measured by D;
and, as many times as D measures A, so many units let there be in E.

Since then D measures A according to the units in E, therefore E by multiplying D has made A. [VII. Def. 15]

And, since A by multiplying B has made C, and A is the product of D, E, therefore the product of D, E by multiplying B has made C.

Therefore C is solid, and D, E, B are its sides.

Q. E. D.

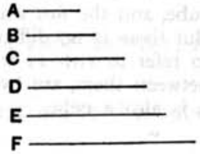
Since a composite number is the product of two factors, the result of multiplying it by another number is to produce a number which is the product of three factors, i.e. a "solid number."

Proposition 8.

If as many numbers as we please beginning from an unit be in continued proportion, the third from the unit will be square, as will also those which successively leave out one; the fourth will be cube, as will also all those which leave out two; and the seventh will be at once cube and square, as will also those which leave out five.

Let there be as many numbers as we please, A, B, C, D, E, F, beginning from an unit and in continued proportion;

I say that B, the third from the unit, is square, as are also all those which leave out one; C, the fourth, is cube, as are also all those which leave out two; and F, the seventh, is at once cube and square, as are also all those which leave out five.



For since, as the unit is to A, so is A to B, therefore the unit measures the number A the same number of times that A measures B. [VII. Def. 20]

But the unit measures the number A according to the units in it;

therefore A also measures B according to the units in A.

Therefore A by multiplying itself has made B;

therefore B is square.

And, since B, C, D are in continued proportion, and B is square,

therefore D is also square. [VIII. 22]

For the same reason
F is also square.

Similarly we can prove that all those which leave out one are square.

I say next that C, the fourth from the unit, is cube, as are also all those which leave out two.

For since, as the unit is to A, so is B to C,
therefore the unit measures the number A the same number of times that B measures C.

But the unit measures the number A according to the units in A;
therefore B also measures C according to the units in A.

Therefore A by multiplying B has made C.

Since then A by multiplying itself has made B, and by multiplying B has made C,
therefore C is cube.

And, since C, D, E, F are in continued proportion, and C is cube,
therefore F is also cube. [VIII. 23]

But it was also proved square;
therefore the seventh from the unit is both cube and square.

Similarly we can prove that all the numbers which leave out five are also both cube and square.

Q. E. D.

If $1, a, a_2, a_3, \ldots$ be a geometrical progression, then $a_2, a_4, a_6, \ldots$ are squares;
$a_3, a_6, a_9, \ldots$ are cubes;
$a_6, a_{12}, \ldots$ are both squares and cubes.

Since $$1 : a = a : a_2,$$
$$a_2 = a^2.$$

And, since a_2, a_3, a_4 are in geometrical progression and $a_2\ (= a^2)$ is a square,
$$a_4 \text{ is a square.} \qquad \text{[VIII. 22]}$$

Similarly $a_6, a_8, \ldots$ are squares.

Next, $$1 : a = a_2 : a_3$$
$$= a^2 : a_3,$$
whence $a_3 = a^3$, a cube number.

And, since a_3, a_4, a_5, a_6 are in geometrical progression, and a_3 is a cube,
$$a_6 \text{ is a cube.} \qquad \text{[VIII. 23]}$$

Similarly a_9, a_{12}, ... are cubes.

Clearly then a_6, a_{12}, a_{18}, ... are both squares and cubes.

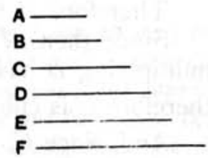
The whole result is of course obvious if the geometrical progression is written, with our notation, as

$$1, a, a^2, a^3, a^4, \ldots a^n.$$

PROPOSITION 9.

If as many numbers as we please beginning from an unit be in continued proportion, and the number after the unit be square, all the rest will also be square. And, if the number after the unit be cube, all the rest will also be cube.

Let there be as many numbers as we please, A, B, C, D, E, F, beginning from an unit and in continued proportion, and let A, the number after the unit, be square;



I say that all the rest will also be square.

Now it has been proved that B, the third from the unit, is square, as are also all those which leave out one; [IX. 8]

I say that all the rest are also square.

For, since A, B, C are in continued proportion, and A is square,

therefore C is also square. [VIII. 22]

Again, since B, C, D are in continued proportion, and B is square,

D is also square. [VIII. 22]

Similarly we can prove that all the rest are also square.

Next, let A be cube;

I say that all the rest are also cube.

Now it has been proved that C, the fourth from the unit, is cube, as also are all those which leave out two; [IX. 8]

I say that all the rest are also cube.

For, since, as the unit is to A, so is A to B, therefore the unit measures A the same number of times as A measures B.

But the unit measures A according to the units in it; therefore A also measures B according to the units in itself; therefore A by multiplying itself has made B.

And A is cube.

But, if a cube number by multiplying itself make some number, the product is cube. [IX. 3]

Therefore B is also cube.

And, since the four numbers A, B, C, D are in continued proportion,
and A is cube,
D also is cube. [VIII. 23]

For the same reason
E is also cube, and similarly all the rest are cube.

Q. E. D.

If $1, a^2, a_2, a_3, a_4, \ldots$ are in geometrical progression, $a_2, a_3, a_4, \ldots$ are all squares;

and, if $1, a^3, a_2, a_3, a_4, \ldots$ are in geometrical progression, $a_2, a_3, \ldots$ are all cubes.

(1) By IX. 8, $a_2, a_4, a_6, \ldots$ are all squares.

And, a^2, a_2, a_3 being in geometrical progression, and a^2 being a square,

a_3 is a square. [VIII. 22]

For the same reason $a_5, a_7, \ldots$ are all squares.

(2) By IX. 8, $a_3, a_6, a_9, \ldots$ are all cubes.

Now $1 : a^3 = a^3 : a_2$.

Therefore $a_2 = a^3 \cdot a^3$, **which is a cube, by IX. 3.**

And, a^3, a_2, a_3, a_4 being in geometrical progression, and a^3 being cube,

a_4 is cube. [VIII. 23]

Similarly we prove that a_5 is cube, and so on.

The results are of course obvious in our notation, the series being

(1) $1, a^2, a^4, a^6, \ldots a^{2n}$,

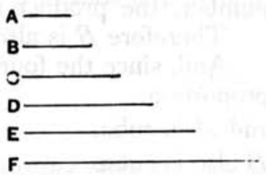
(2) $1, a^3, a^9, a^{12}, \ldots a^{3n}$.

PROPOSITION 10.

If as many numbers as we please beginning from an unit be in continued proportion, and the number after the unit be not square, neither will any other be square except the third from the unit and all those which leave out one. And, if the number after the unit be not cube, neither will any other be cube except the fourth from the unit and all those which leave out two.

Let there be as many numbers as we please, A, B, C, D, E, F, beginning from an unit and in continued proportion, and let A, the number after the unit, not be square;

I say that neither will any other be square except the third from the unit <and those which leave out one>.



For, if possible, let C be square.
But B is also square; [IX. 8]
[therefore B, C have to one another the ratio which a square number has to a square number].

And, as B is to C, so is A to B;
therefore A, B have to one another the ratio which a square number has to a square number;
[so that A, B are similar plane numbers]. [VIII. 26, converse]

And B is square;
therefore A is also square:
which is contrary to the hypothesis.

Therefore C is not square.

Similarly we can prove that neither is any other of the numbers square except the third from the unit and those which leave out one.

Next, let A not be cube.

I say that neither will any other be cube except the fourth from the unit and those which leave out two.

For, if possible, let D be cube.
Now C is also cube; for it is fourth from the unit. [IX. 8]
And, as C is to D, so is B to C;
therefore B also has to C the ratio which a cube has to a cube.

And C is cube;
therefore B is also cube. [VIII. 25]

And since, as the unit is to A, so is A to B,
and the unit measures A according to the units in it,
therefore A also measures B according to the units in itself;
therefore A by multiplying itself has made the cube number B.

But, if a number by multiplying itself make a cube number, it is also itself cube. [IX. 6]

Therefore A is also cube:
which is contrary to the hypothesis.

Therefore D is not cube.

Similarly we can prove that neither is any other of the numbers cube except the fourth from the unit and those which leave out two.

Q. E. D.

If 1, a, a_2, a_3, a_4, ... be a geometrical progression, then (1), if a is not a square, none of the terms will be square except a_2, a_4, a_6, ...;
and (2), if a is not a cube, none of the terms will be cube except a_3, a_6, a_9, ...

With reference to the first part of the proof, viz. that which proves that, if a_3 is a square, a must be a square, Heiberg remarks that the words which I have bracketed are perhaps spurious; for it is easier to use VIII. 24 than the *converse* of VIII. 26, and a use of VIII. 24 would correspond better to the use of VIII. 25 in the second part relating to cubes. I agree in this view and have bracketed the words accordingly. (See however note, p. 383, on converses of VIII. 26, 27 given by Heron.) If this change be made, the proof runs as follows.

(1) If possible, let a_3 be square.

Now $$a_2 : a_3 = a : a_2.$$

But a_2 is a square. [IX. 8]

Therefore a is to a_2 in the ratio of a square to a square.

And a_2 is square;

therefore a is square [VIII. 24]: which is impossible.

(2) If possible, let a_4 be a cube.

Now $$a_3 : a_4 = a_2 : a_3.$$

And a_3 is a cube. [IX. 8]

Therefore a_2 is to a_3 in the ratio of a cube to a cube.

And a_3 is a cube:

therefore a_2 is a cube. [VIII. 25]

But, since $$1 : a = a : a_2,$$
$$a_2 = a^2.$$

And, since a^2 is a cube,

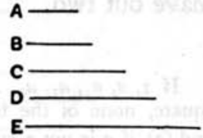
a must be a cube [IX. 6]: which is impossible.

The propositions VIII. 24, 25 are here not quoted in their exact form in that the *first* and *second* squares, or cubes, change places. But there is no difficulty, since the method by which the theorems are proved shows that either inference is equally correct.

PROPOSITION 11.

If as many numbers as we please beginning from an unit be in continued proportion, the less measures the greater according to some one of the numbers which have place among the proportional numbers.

Let there be as many numbers as we please, B, C, D, E, beginning from the unit A and in continued proportion;



I say that B, the least of the numbers B, C, D, E, measures E according to some one of the numbers C, D.

For since, as the unit A is to B, so is D to E,

therefore the unit A measures the number B the same number of times as D measures E;

therefore, alternately, the unit A measures D the same number of times as B measures E. [VII. 15]

But the unit A measures D according to the units in it;

therefore B also measures E according to the units in D;

so that B the less measures E the greater according to some number of those which have place among the proportional numbers.—

PORISM. And it is manifest that, whatever place the measuring number has, reckoned from the unit, the same place also has the number according to which it measures, reckoned from the number measured, in the direction of the number before it.—

Q. E. D.

The proposition and the porism together assert that, if $1, a, a_2, \ldots a_n$ be a geometrical progression, a_r measures a_n and gives the quotient a_{n-r} $(r < n)$.

Euclid only proves that $a_n = a \cdot a_{n-1}$, as follows.

$$1 : a = a_{n-1} : a_n.$$

Therefore 1 measures a the same number of times as a_{n-1} measures a_n.

Hence 1 measures a_{n-1} the same number of times as a measures a_n; [VII. 15]

that is, $$a_n = a \cdot a_{n-1}.$$

We can supply the proof of the porism as follows.

$$1 : a = a_r : a_{r+1},$$
$$a : a_2 = a_{r+1} : a_{r+2},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$a_{n-r-1} : a_{n-r} = a_{n-1} : a_n,$$

whence, *ex aequali*,

$$1 : a_{n-r} = a_r : a_n. \quad \text{[VII. 14]}$$

It follows, by the same argument as before, that

$$a_n = a_r \cdot a_{n-r}.$$

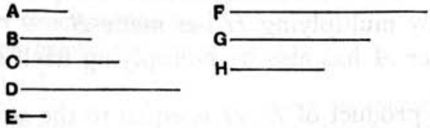
With our notation, we have the theorem of indices that

$$a^{m+n} = a^m \cdot a^n.$$

PROPOSITION 12.

If as many numbers as we please beginning from an unit be in continued proportion, by however many prime numbers the last is measured, the next to the unit will also be measured by the same.

Let there be as many numbers as we please, A, B, C, D, beginning from an unit, and in continued proportion;
I say that, by however many prime numbers D is measured, A will also be measured by the same.



For let D be measured by any prime number E;
I say that E measures A.

For suppose it does not;
now E is prime, and any prime number is prime to any which it does not measure; [VII. 29]
therefore E, A are prime to one another.

And, since E measures D, let it measure it according to F,
therefore E by multiplying F has made D.

Again, since A measures D according to the units in C, [IX. 11 and Por.]
therefore A by multiplying C has made D.

But, further, E has also by multiplying F made D;
therefore the product of A, C is equal to the product of E, F.

Therefore, as A is to E, so is F to C. [VII. 19]

But A, E are prime,
primes are also least, [VII. 21]
and the least measure those which have the same ratio the same number of times, the antecedent the antecedent and the consequent the consequent; [VII. 20]
therefore E measures C.

Let it measure it according to G;
therefore E by multiplying G has made C.

But, further, by the theorem before this,
A has also by multiplying B made C. [IX. 11 and Por.]

Therefore the product of A, B is equal to the product of E, G.

Therefore, as A is to E, so is G to B. [VII. 19]

But A, E are prime,
primes are also least, [VII. 21]
and the least numbers measure those which have the same ratio with them the same number of times, the antecedent the antecedent and the consequent the consequent: [VII. 20]
therefore E measures B.

Let it measure it according to H;
therefore E by multiplying H has made B.

But further A has also by multiplying itself made B; [IX. 8]
therefore the product of E, H is equal to the square on A.

Therefore, as E is to A, so is A to H. [VII. 19]

But A, E are prime,
primes are also least, [VII. 21]
and the least measure those which have the same ratio the same number of times, the antecedent the antecedent and the consequent the consequent; [VII. 20]
therefore E measures A, as antecedent antecedent.

But, again, it also does not measure it:
which is impossible.

Therefore E, A are not prime to one another.

Therefore they are composite to one another.

But numbers composite to one another are measured by some number. [VII. Def. 14]

And, since E is by hypothesis prime,
and the prime is not measured by any number other than itself,
therefore E measures A, E,
so that E measures A.

[But it also measures D;
therefore E measures A, D.]

Similarly we can prove that, by however many prime numbers D is measured, A will also be measured by the same.

Q. E. D.

If $1, a, a_2, \ldots a_n$ be a geometrical progression, and a_n be measured by any prime number p, a will also be measured by p.

For, if possible, suppose that p does not measure a; then, p being prime, p, a are prime to one another. [VII. 29]

Suppose $a_n = m \,.\, p$.

Now $a_n = a \,.\, a_{n-1}$. [IX. 11]

Therefore $a \,.\, a_{n-1} = m \,.\, p$,

and $a : p = m : a_{n-1}$. [VII. 19]

Hence, a, p being prime to one another,

p measures a_{n-1}. [VII. 20, 21]

By a repetition of the same process, we can prove that p measures a_{n-2} and therefore a_{n-3}, and so on, and finally that p measures a.

But, by hypothesis, p does not measure a: which is impossible.

Hence p, a are not prime to one another:

therefore they have some common factor. [VII. Def. 14]

But p is the only number which measures p;

therefore p measures a.

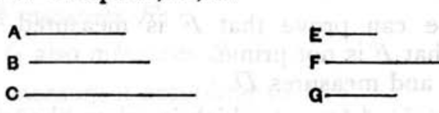
Heiberg remarks that, as, in the ἔκθεσις, Euclid sets himself to prove that E measures A, the words bracketed above are unnecessary and therefore perhaps interpolated.

Proposition 13.

If as many numbers as we please beginning from an unit be in continued proportion, and the number after the unit be prime, the greatest will not be measured by any except those which have a place among the proportional numbers.

Let there be as many numbers as we please, A, B, C, D, beginning from an unit and in continued proportion, and let A, the number after the unit, be prime;

I say that D, the greatest of them, will not be measured by any other number except A, B, C.



For, if possible, let it be measured by E, and let E not be the same with any of the numbers A, B, C.

It is then manifest that E is not prime.

For, if E is prime and measures D,

it will also measure A [IX. 12], which is prime, though it is not the same with it:

which is impossible.

Therefore E is not prime.

Therefore it is composite.

But any composite number is measured by some prime number; [VII. 31]

therefore E is measured by some prime number.

I say next that it will not be measured by any other prime except A.

For, if E is measured by another,
and E measures D,
that other will also measure D;
so that it will also measure A [IX. 12], which is prime, though it is not the same with it:
which is impossible.

Therefore A measures E.

And, since E measures D, let it measure it according to F.

I say that F is not the same with any of the numbers A, B, C.

For, if F is the same with one of the numbers A, B, C, and measures D according to E,
therefore one of the numbers A, B, C also measures D according to E.

But one of the numbers A, B, C measures D according to some one of the numbers A, B, C; [IX. 11]
therefore E is also the same with one of the numbers A, B, C:
which is contrary to the hypothesis.

Therefore F is not the same as any one of the numbers $A. B, C$.

Similarly we can prove that F is measured by A, by proving again that F is not prime.

For, if it is, and measures D,
it will also measure A [IX. 12], which is prime, though it is not the same with it:
which is impossible;
therefore F is not prime.

Therefore it is composite.

But any composite number is measured by some prime number; [VII. 31]

therefore F is measured by some prime number.

I say next that it will not be measured by any other prime except A.

For, if any other prime number measures F,
and F measures D,
that other will also measure D;
so that it will also measure A [IX. 12], which is prime, though it is not the same with it:
which is impossible.

Therefore A measures F.

And, since E measures D according to F,
therefore E by multiplying F has made D.

But, further, A has also by multiplying C made D; [IX. 11]
therefore the product of A, C is equal to the product of E, F.

Therefore, proportionally, as A is to E, so is F to C. [VII. 19]

But A measures E;
therefore F also measures C.

Let it measure it according to G.

Similarly, then, we can prove that G is not the same with any of the numbers A, B, and that it is measured by A.

And, since F measures C according to G
therefore F by multiplying G has made C.

But, further, A has also by multiplying B made C; [IX. 11]
therefore the product of A, B is equal to the product of F, G.

Therefore, proportionally, as A is to F, so is G to B. [VII. 19]

But A measures F;
therefore G also measures B.

Let it measure it according to H.

Similarly then we can prove that H is not the same with A.

And, since G measures B according to H,
therefore G by multiplying H has made B.

But further A has also by multiplying itself made B; [IX. 8]
therefore the product of H, G is equal to the square on A.

Therefore, as H is to A, so is A to G. [VII. 19]

But A measures G;
therefore H also measures A, which is prime, though it is not the same with it:
which is absurd.

Therefore D the greatest will not be measured by any other number except A, B, C.

Q. E. D.

If $1, a, a_2, \ldots a_n$ be a geometrical progression, and if a is prime, a_n will not be measured by any numbers except the preceding terms of the series.

If possible, let a_n be measured by b, a number different from all the preceding terms.

Now b cannot be prime, for, if it were, it would measure a. [IX. 12]

Therefore b is composite, and hence will be measured by *some* prime number [VII. 31], say p.

Thus p must measure a_n and therefore a [IX. 12]; so that p cannot be different from a, and b is not measured by any prime number except a.

Suppose that $$a_n = b \, . \, c.$$

Now c cannot be identical with any of the terms $a, a_2, \ldots a_{n-1}$; for, if it were, b would be identical with another of them: [IX. 11]
which is contrary to the hypothesis.

We can now prove (just as for b) that c cannot be prime and cannot be measured by any prime number except a.

Since $$b \, . \, c = a_n = a \, . \, a_{n-1},$$ [IX. 11]
$$a : b = c : a_{n-1},$$
whence, since a measures b,
$$c \text{ measures } a_{n-1}.$$

Let $$a_{n-1} = c \, . \, d.$$

We now prove in the same way that d is not identical with any of the terms $a, a_2, \ldots a_{n-2}$, is not prime, and is not measured by any prime except a, and also that
$$d \text{ measures } a_{n-2}.$$

Proceeding in this way, we get a last factor, say k, which measures a though different from it:
which is absurd, since a is prime.

Thus the original supposition that a_n can be measured by a number b different from all the terms $a, a_2, \ldots a_{n-1}$ must be incorrect.

Therefore etc.

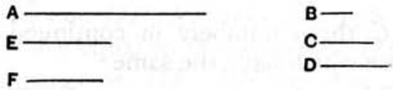
PROPOSITION 14.

If a number be the least that is measured by prime numbers, it will not be measured by any other prime number except those originally measuring it.

For let the number A be the least that is measured by the prime numbers B, C, D;

I say that A will not be measured by any other prime number except B, C, D.

For, if possible, let it be measured by the prime number E, and let E not be the same with any one of the numbers B, C, D.



Now, since E measures A, let it measure it according to F;

therefore E by multiplying F has made A.

And A is measured by the prime numbers B, C, D.

But, if two numbers by multiplying one another make some number, and any prime number measure the product, it will also measure one of the original numbers; [VII. 30]

therefore B, C, D will measure one of the numbers E, F.

Now they will not measure E;

for E is prime and not the same with any one of the numbers B, C, D.

Therefore they will measure F, which is less than A: which is impossible, for A is by hypothesis the least number measured by B, C, D.

Therefore no prime number will measure A except B, C, D.

Q. E. D.

In other words, a number can be resolved into prime factors in only one way.

Let a be the least number measured by each of the prime numbers b, c, d, ... k.

If possible, suppose that a has a prime factor p different from b, c, d, ... k.

Let $$a = p \cdot m.$$

Now b, c, d, ... k, measuring a, must measure one of the two factors p, m. [VII. 30]

They do not, by hypothesis, measure p;

therefore they must measure m, a number less than a:

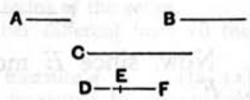
which is contrary to the hypothesis.

Therefore a has no prime factors except b, c, d, ... k.

Proposition 15.

If three numbers in continued proportion be the least of those which have the same ratio with them, any two whatever added together will be prime to the remaining number.

Let A, B, C, three numbers in continued proportion, be the least of those which have the same ratio with them;



I say that any two of the numbers A, B, C whatever added together are prime to the remaining number, namely A, B to C; B, C to A; and further A, C to B.

For let two numbers DE, EF, the least of those which have the same ratio with A, B, C, be taken. [VIII. 2]

It is then manifest that DE by multiplying itself has made A, and by multiplying EF has made B, and, further, EF by multiplying itself has made C. [VIII. 2]

Now, since DE, EF are least,

they are prime to one another. [VII. 22]

But, if two numbers be prime to one another,

their sum is also prime to each; [VII. 28]

therefore DF is also prime to each of the numbers DE, EF.

But further DE is also prime to EF;

therefore DF, DE are prime to EF.

But, if two numbers be prime to any number,

their product is also prime to the other; [VII. 24]

so that the product of FD, DE is prime to EF;

hence the product of FD, DE is also prime to the square on EF. [VII. 25]

But the product of FD, DE is the square on DE together with the product of DE, EF; [II. 3]

therefore the square on DE together with the product of DE, EF is prime to the square on EF.

And the square on DE is A,

the product of DE, EF is B,

and the square on EF is C;

therefore A, B added together are prime to C.

Similarly we can prove that B, C added together are prime to A.

I say next that A, C added together are also prime to B.

For, since DF is prime to each of the numbers DE, EF, the square on DF is also prime to the product of DE, EF. [VII. 24, 25]

But the squares on DE, EF together with twice the product of DE, EF are equal to the square on DF; [II. 4] therefore the squares on DE, EF together with twice the product of DE, EF are prime to the product of DE, EF.

Separando, the squares on DE, EF together with once the product of DE, EF are prime to the product of DE, EF.

Therefore, *separando* again, the squares on DE, EF are prime to the product of DE, EF.

And the square on DE is A,
the product of DE, EF is B,
and the square on EF is C.

Therefore A, C added together are prime to B.

Q. E. D.

If a, b, c be a geometrical progression in the least terms which have a given common ratio, $(b+c)$, $(c+a)$, $(a+b)$ are respectively prime to a, b, c.

Let $\alpha : \beta$ be the common ratio in its lowest terms, so that the geometrical progression is

$$\alpha^2, \quad \alpha\beta, \quad \beta^2. \qquad \text{[VIII. 2]}$$

Now, α, β being prime to one another,
$\alpha + \beta$ is prime to both α and β. [VII. 28]

Therefore $(\alpha+\beta)$, α are both prime to β.

Hence $(\alpha+\beta)\,\alpha$ is prime to β, [VII. 24]
and therefore to β^2; [VII. 25]

i.e. $\alpha^2 + \alpha\beta$ is prime to β^2,
or $a + b$ is prime to c.

Similarly, $\alpha\beta + \beta^2$ is prime to α^2,
or $b + c$ is prime to a.

Lastly, $\alpha + \beta$ being prime to both α and β,
$(\alpha+\beta)^2$ is prime to $\alpha\beta$, [VII. 24, 25]
or $\alpha^2 + \beta^2 + 2\alpha\beta$ is prime to $\alpha\beta$:
whence $\alpha^2 + \beta^2$ is prime to $\alpha\beta$.

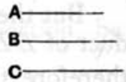
The latter inference, made in two steps, may be proved by *reductio ad absurdum* as Commandinus proves it.

If $\alpha^2 + \beta^2$ is not prime to $\alpha\beta$, let x measure them;
therefore x measures $\alpha^2 + \beta^2 + 2\alpha\beta$ as well as $\alpha\beta$;
hence $\alpha^2 + \beta^2 + 2\alpha\beta$ and $\alpha\beta$ are not prime to one another, which is contrary to the hypothesis.

Proposition 16.

If two numbers be prime to one another, the second will not be to any other number as the first is to the second.

For let the two numbers A, B be prime to one another; I say that B is not to any other number as A is to B.



For, if possible, as A is to B, so let B be to C.

Now A, B are prime,

primes are also least, [VII. 21]

and the least numbers measure those which have the same ratio the same number of times, the antecedent the antecedent and the consequent the consequent; [VII. 20]

therefore A measures B as antecedent antecedent.

But it also measures itself;

therefore A measures A, B which are prime to one another: which is absurd.

Therefore B will not be to C, as A is to B.

Q. E. D.

If a, b are prime to one another, they can have no integral third proportional.

If possible, let $a : b = b : x$.

Therefore [VII. 20, 21] a measures b; and a, b have the common measure a, which is contrary to the hypothesis.

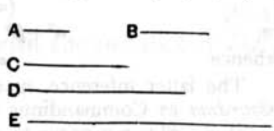
Proposition 17.

If there be as many numbers as we please in continued proportion, and the extremes of them be prime to one another, the last will not be to any other number as the first to the second.

For let there be as many numbers as we please, A, B, C, D, in continued proportion,

and let the extremes of them, A, D, be prime to one another;

I say that D is not to any other number as A is to B.



For, if possible, as A is to B, so let D be to E;

therefore, alternately, as A is to D, so is B to E. [VII. 13]

But A, D are prime,
primes are also least, [VII. 21]
and the least numbers measure those which have the same ratio the same number of times, the antecedent the antecedent and the consequent the consequent. [VII. 20]

Therefore A measures B.

And, as A is to B, so is B to C.

Therefore B also measures C;
so that A also measures C.

And since, as B is to C, so is C to D,
and B measures C,
therefore C also measures D.

But A measured C;
so that A also measures D.

But it also measures itself;
therefore A measures A, D which are prime to one another: which is impossible.

Therefore D will not be to any other number as A is to B.

Q. E. D.

If $a, a_2, a_3, \dots a_n$ be a geometrical progression, and a, a_n are prime to one another, then a, a_2, a_n can have no integral fourth proportional.

For, if possible, let $a : a_2 = a_n : x$.

Therefore $a : a_n = a_2 : x$,

and hence [VII. 20, 21] a measures a_2.

Therefore a_2 measures a_3, [VII. Def. 20]

and hence a measures a_3, and therefore also ultimately a_n.

Thus a, a_n are both measured by a: which is contrary to the hypothesis.

PROPOSITION 18.

Given two numbers, to investigate whether it is possible to find a third proportional to them.

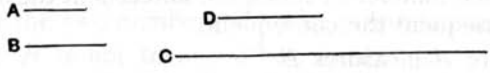
Let A, B be the given two numbers, and let it be required to investigate whether it is possible to find a third proportional to them.

Now A, B are either prime to one another or not.

And, if they are prime to one another, it has been proved that it is impossible to find a third proportional to them. [IX. 16]

Next, let A, B not be prime to one another,
and let B by multiplying itself make C.

Then A either measures C or does not measure it.



First, let it measure it according to D;
therefore A by multiplying D has made C.

But, further, B has also by multiplying itself made C;
therefore the product of A, D is equal to the square on B.

Therefore, as A is to B, so is B to D; [VII. 19]
therefore a third proportional number D has been found to A, B.

Next, let A not measure C;
I say that it is impossible to find a third proportional number to A, B.

For, if possible, let D, such third proportional, have been found.

Therefore the product of A, D is equal to the square on B.

But the square on B is C;
therefore the product of A, D is equal to C.

Hence A by multiplying D has made C;
therefore A measures C according to D.

But, by hypothesis, it also does not measure it:
which is absurd.

Therefore it is not possible to find a third proportional number to A, B when A does not measure C. Q. E. D.

Given two numbers a, b, to find the condition that they may have an integral third proportional.

(1) a, b must not be prime to one another. [IX. 16]

(2) a must measure b^2.

For, if a, b, c be in continued proportion,

$$ac = b^2.$$

Therefore a measures b^2.

Condition (1) is included in condition (2) since, if $b^2 = ma$, a and b cannot be prime to one another.

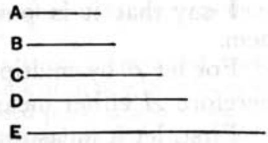
The result is of course easily seen if the three terms in continued proportion be written

$$a, \quad a\frac{b}{a}, \quad a\left(\frac{b}{a}\right)^2.$$

PROPOSITION 19.

Given three numbers, to investigate when it is possible to find a fourth proportional to them.

Let A, B, C be the given three numbers, and let it be required to investigate when it is possible to find a fourth proportional to them.



Now either they are not in continued proportion, and the extremes of them are prime to one another;

or they are in continued proportion, and the extremes of them are not prime to one another;

or they are not in continued proportion, nor are the extremes of them prime to one another;

or they are in continued proportion, and the extremes of them are prime to one another.

If then A, B, C are in continued proportion, and the extremes of them A, C are prime to one another, it has been proved that it is impossible to find a fourth proportional number to them. [IX. 17]

†Next, let A, B, C not be in continued proportion, the extremes being again prime to one another;

I say that in this case also it is impossible to find a fourth proportional to them.

For, if possible, let D have been found, so that,

as A is to B, so is C to D,

and let it be contrived that, as B is to C, so is D to E.

Now, since, as A is to B, so is C to D,

and, as B is to C, so is D to E,

therefore, *ex aequali*, as A is to C, so is C to E. [VII. 14]

But A, C are prime,

primes are also least, [VII. 21]

and the least numbers measure those which have the same ratio, the antecedent the antecedent and the consequent the consequent. [VII. 20]

Therefore A measures C as antecedent antecedent.

But it also measures itself;
therefore A measures A, C which are prime to one another: which is impossible.

Therefore it is not possible to find a fourth proportional to A, B, C.†

Next, let A, B, C be again in continued proportion, but let A, C not be prime to one another.

I say that it is possible to find a fourth proportional to them.

For let B by multiplying C make D;
therefore A either measures D or does not measure it.

First, let it measure it according to E;
therefore A by multiplying E has made D.

But, further, B has also by multiplying C made D;
therefore the product of A, E is equal to the product of B, C;
therefore, proportionally, as A is to B, so is C to E; [VII. 19]
therefore E has been found a fourth proportional to A, B, C.

Next, let A not measure D;
I say that it is impossible to find a fourth proportional number to A, B, C.

For, if possible, let E have been found;
therefore the product of A, E is equal to the product of B, C. [VII. 19]

But the product of B, C is D;
therefore the product of A, E is also equal to D.

Therefore A by multiplying E has made D;
therefore A measures D according to E,
so that A measures D.

But it also does not measure it:
which is absurd.

Therefore it is not possible to find a fourth proportional number to A, B, C when A does not measure D.

Next, let A, B, C not be in continued proportion, nor the extremes prime to one another.

And let B by multiplying C make D.

Similarly then it can be proved that, if A measures D, it is possible to find a fourth proportional to them, but, if it does not measure it, impossible. Q. E. D.

Given three numbers a, b, c, to find the condition that they may have an integral fourth proportional.

The Greek text of part of this proposition is hopelessly corrupt. According to it Euclid takes four cases.

(1) a, b, c not in continued proportion, and a, c prime to one another.

(2) a, b, c in continued proportion, and a, c not prime to one another.

(3) a, b, c not in continued proportion, and a, c not prime to one another.

(4) a, b, c in continued proportion, and a, c prime to one another.

(4) is the case dealt with in IX. 17, where it is shown that on hypothesis (4) a fourth proportional cannot be found.

The text now takes case (1) and asserts that a fourth proportional cannot be found in this case either. We have only to think of 4, 6, 9 in order to see that there is something wrong here. The supposed proof is also wrong. If possible, says the text, let d be a fourth proportional to a, b, c, *and let* e *be taken such that*

$$b : c = d : e.$$

Then, *ex aequali*, $$a : c = c : e,$$

whence a measures c: [VII. 20, 21]

which is impossible, since a, c are prime to one another.

But this does not prove that a fourth proportional d cannot be found; it only proves that, if d is a fourth proportional, no integer e can be found to satisfy the equation

$$b : c = d : e.$$

Indeed it is obvious from IX. 16 that in the equation

$$a : c = c : e$$

e cannot be integral.

The cases (2) and (3) are correctly given, the first in full, and the other as a case to be proved "similarly" to it.

These two cases really give all that is necessary.

Let the product bc be taken.

Then, if a measures bc, suppose $bc = ad$;

therefore $$a : b = c : d,$$

and d is a fourth proportional.

But, if a does *not* measure bc, no fourth proportional can be found. For, if x were a fourth proportional, ax would be equal to bc, and a would measure bc.

The sufficient condition in any case for the possibility of finding a fourth proportional to a, b, c is that a should measure bc.

Theon appears to have corrected the proof by leaving out the incorrect portion which I have included between daggers and the last case (3) dealt with in the last lines. Also, in accordance with this arrangement, he does not distinguish four cases at the beginning but only two. "Either A, B, C are in continued proportion and the extremes of them A, C are prime to one another; or not." Then, instead of introducing case (2) by the words "Next let A, B, C...to find a fourth proportional to them," immediately following the second dagger above, Theon merely says "*But, if not,*" [i.e. if it is not the case that a, b, c are in G.P. and a, c prime to one another] "let B by multiplying C make D," and so on.

August adopts Theon's form of the proof. Heiberg does not feel able to do this, in view of the superiority of the authority for the text as given above (P); he therefore retains the latter without any attempt to emend it.

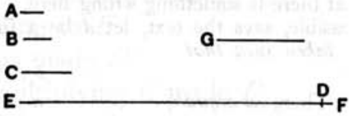
Proposition 20.

Prime numbers are more than any assigned multitude of prime numbers.

Let A, B, C be the assigned prime numbers;

I say that there are more prime numbers than A, B, C.

For let the least number measured by A, B, C be taken,



and let it be DE;

let the unit DF be added to DE.

Then EF is either prime or not.

First, let it be prime;

then the prime numbers A, B, C, EF have been found which are more than A, B, C.

Next, let EF not be prime;

therefore it is measured by some prime number. [VII. 31]

Let it be measured by the prime number G.

I say that G is not the same with any of the numbers A, B, C.

For, if possible, let it be so.

Now A, B, C measure DE;

therefore G also will measure DE.

But it also measures EF.

Therefore G, being a number, will measure the remainder, the unit DF:

which is absurd.

Therefore G is not the same with any one of the numbers A, B, C.

And by hypothesis it is prime.

Therefore the prime numbers A, B, C, G have been found which are more than the assigned multitude of A, B, C.

Q. E. D.

We have here the important proposition that *the number of prime numbers is infinite.*

The proof will be seen to be the same as that given in our algebraical text-books. Let $a, b, c, \ldots k$ be any prime numbers.

Take the product $abc \ldots k$ and add unity.

Then $(abc \ldots k + 1)$ is either a prime number or not a prime number.

(1) If it *is*, we have added another prime number to those given.

(2) If it is *not*, it must be measured by some prime number [VII. 31], say p.

Now p cannot be identical with any of the prime numbers $a, b, c, \ldots k$.

For, if it is, it will divide $abc \ldots k$.

Therefore, since it divides $(abc \ldots k + 1)$ also, it will measure the difference, or unity:

which is impossible.

Therefore in any case we have obtained one fresh prime number.

And the process can be carried on to any extent.

PROPOSITION 21.

If as many even numbers as we please be added together, the whole is even.

For let as many even numbers as we please, AB, BC, CD, DE, be added together;

I say that the whole AE is even.

For, since each of the numbers AB, BC, CD, DE is even, it has a half part; [VII. Def. 6]

so that the whole AE also has a half part.

But an even number is that which is divisible into two equal parts; [*id.*]

therefore AE is even.

Q. E. D.

In this and the following propositions up to IX. 34 inclusive we have a number of theorems about odd, even, "even-times even" and "even-times odd" numbers respectively. They are all simple and require no explanation in order to enable them to be followed easily.

PROPOSITION 22.

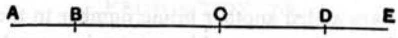
If as many odd numbers as we please be added together, and their multitude be even, the whole will be even.

For let as many odd numbers as we please, AB, BC, CD, DE, even in multitude, be added together;

I say that the whole AE is even.

For, since each of the numbers AB, BC, CD, DE is odd, if an unit be subtracted from each, each of the remainders will be even; [VII. Def. 7]

so that the sum of them will be even. [IX. 21]



But the multitude of the units is also even.

Therefore the whole AE is also even. [IX. 21]

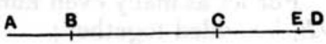
Q. E. D.

Proposition 23.

If as many odd numbers as we please be added together, and their multitude be odd, the whole will also be odd.

For let as many odd numbers as we please, AB, BC, CD, the multitude of which is odd, be added together;

I say that the whole AD is also odd.



Let the unit DE be subtracted from CD;

therefore the remainder CE is even. [VII. Def. 7]

But CA is also even; [IX. 22]

therefore the whole AE is also even. [IX. 21]

And DE is an unit.

Therefore AD is odd. [VII. Def. 7]

Q. E. D.

3. Literally "let there be as many numbers as we please, of which *let* the multitude *be* odd." This form, natural in Greek, is awkward in English.

Proposition 24.

If from an even number an even number be subtracted, the remainder will be even.

For from the even number AB let the even number BC be subtracted:

I say that the remainder CA is even.

For, since AB is even, it has a half part. [VII. Def. 6]

For the same reason BC also has a half part;
so that the remainder [CA also has a half part, and] AC is therefore even.

Q. E. D.

Proposition 25.

If from an even number an odd number be subtracted, the remainder will be odd.

For from the even number AB let the odd number BC be subtracted;

I say that the remainder CA is odd.

For let the unit CD be subtracted from BC;

therefore DB is even. [VII. Def. 7]

But AB is also even;

therefore the remainder AD is also even. [IX. 24]

And CD is an unit;

therefore CA is odd. [VII. Def. 7]

Q. E. D.

Proposition 26.

If from an odd number an odd number be subtracted, the remainder will be even.

For from the odd number AB let the odd number BC be subtracted;

I say that the remainder CA is even.

For, since AB is odd, let the unit BD be subtracted;

therefore the remainder AD is even. [VII. Def. 7]

For the same reason CD is also even; [VII. Def. 7]

so that the remainder CA is also even. [IX. 24]

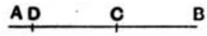
Q. E. D.

PROPOSITION 27.

If from an odd number an even number be subtracted, the remainder will be odd.

For from the odd number AB let the even number BC be subtracted;

I say that the remainder CA is odd.



Let the unit AD be subtracted;

therefore DB is even. [VII. Def. 7]

But BC is also even;

therefore the remainder CD is even. [IX. 24]

Therefore CA is odd. [VII. Def. 7]

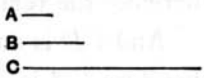
Q. E. D.

PROPOSITION 28.

If an odd number by multiplying an even number make some number, the product will be even.

For let the odd number A by multiplying the even number B make C;

I say that C is even.



For, since A by multiplying B has made C,

therefore C is made up of as many numbers equal to B as there are units in A. [VII. Def. 15]

And B is even;

therefore C is made up of even numbers.

But, if as many even numbers as we please be added together, the whole is even. [IX. 21]

Therefore C is even.

Q. E. D.

PROPOSITION 29.

If an odd number by multiplying an odd number make some number, the product will be odd.

For let the odd number A by multiplying the odd number B make C;

I say that C is odd.

For, since A by multiplying B has made C, therefore C is made up of as many numbers equal to B as there are units in A. [VII. Def. 15]

And each of the numbers A, B is odd;

therefore C is made up of odd numbers the multitude of which is odd.

Thus C is odd. [IX. 23]

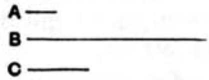
Q. E. D.

Proposition 30.

If an odd number measure an even number, it will also measure the half of it.

For let the odd number A measure the even number B;

I say that it will also measure the half of it.



For, since A measures B,

let it measure it according to C;

I say that C is not odd.

For, if possible, let it be so.

Then, since A measures B according to C,

therefore A by multiplying C has made B.

Therefore B is made up of odd numbers the multitude of which is odd.

Therefore B is odd: [IX. 23]

which is absurd, for by hypothesis it is even.

Therefore C is not odd;

therefore C is even.

Thus A measures B an even number of times.

For this reason then it also measures the half of it.

Q. E. D.

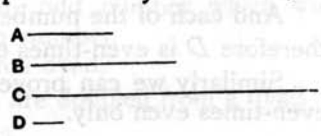
Proposition 31.

If an odd number be prime to any number, it will also be prime to the double of it.

For let the odd number A be prime to any number B,

and let C be double of B;

I say that A is prime to C.



For, if they are not prime to one another, some number will measure them.

Let a number measure them, and let it be D.

Now A is odd;
therefore D is also odd.

And since D which is odd measures C,
and C is even,
therefore [D] will measure the half of C also. [IX. 30]

But B is half of C;
therefore D measures B.

But it also measures A;
therefore D measures A, B which are prime to one another: which is impossible.

Therefore A cannot but be prime to C.

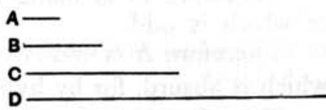
Therefore A, C are prime to one another.

Q. E. D.

Proposition 32.

Each of the numbers which are continually doubled beginning from a dyad is even-times even only.

For let as many numbers as we please, B, C, D, have been continually doubled beginning from the dyad A;
I say that B, C, D are even-times even only.



Now that each of the numbers B, C, D is even-times even is manifest; for it is doubled from a dyad.

I say that it is also even-times even only.

For let an unit be set out.

Since then as many numbers as we please beginning from an unit are in continued proportion,
and the number A after the unit is prime,
therefore D, the greatest of the numbers A, B, C, D, will not be measured by any other number except A, B, C. [IX. 13]

And each of the numbers A, B, C is even;
therefore D is even-times even only. [VII. Def. 8]

Similarly we can prove that each of the numbers B, C is even-times even only.

Q. E. D.

See the notes on VII. Deff. 8 to 11 for a discussion of the difficulties shown by Iamblichus to be involved by the Euclidean definitions of "even-times even," "even-times odd" and "odd-times even."

PROPOSITION 33.

If a number have its half odd, it is even-times odd only.

For let the number A have its half odd;

I say that A is even-times odd only.

Now that it is even-times odd is manifest; for the half of it, being odd, measures it an even number of times. [VII. Def. 9]

I say next that it is also even-times odd only.

For, if A is even-times even also,

it will be measured by an even number according to an even number; [VII. Def. 8]

so that the half of it will also be measured by an even number though it is odd:

which is absurd.

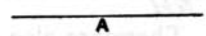
Therefore A is even-times odd only. Q. E. D.

PROPOSITION 34.

If a number neither be one of those which are continually doubled from a dyad, nor have its half odd, it is both even-times even and even-times odd.

For let the number A neither be one of those doubled from a dyad, nor have its half odd;

I say that A is both even-times even and even-times odd.



Now that A is even-times even is manifest;

for it has not its half odd. [VII. Def. 8]

I say next that it is also even-times odd.

For, if we bisect A, then bisect its half, and do this continually, we shall come upon some odd number which will measure A according to an even number.

For, if not, we shall come upon a dyad,

and A will be among those which are doubled from a dyad:

which is contrary to the hypothesis.

Thus A is even-times odd.

But it was also proved even-times even.

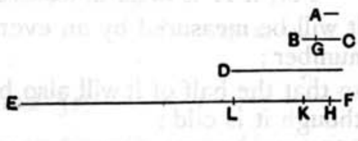
Therefore A is both even-times even and even-times odd.

Q. E. D.

Proposition 35.

If as many numbers as we please be in continued proportion, and there be subtracted from the second and the last numbers equal to the first, then, as the excess of the second is to the first, so will the excess of the last be to all those before it.

Let there be as many numbers as we please in continued proportion, A, BC, D, EF, beginning from A as least, and let there be subtracted from BC and EF the numbers BG, FH, each equal to A; I say that, as GC is to A, so is EH to A, BC, D.



For let FK be made equal to BC, and FL equal to D.

Then, since FK is equal to BC, and of these the part FH is equal to the part BG, therefore the remainder HK is equal to the remainder GC.

And since, as EF is to D, so is D to BC, and BC to A, while D is equal to FL, BC to FK, and A to FH, therefore, as EF is to FL, so is LF to FK, and FK to FH.

Separando, as EL is to LF, so is LK to FK, and KH to FH. [VII. 11, 13]

Therefore also, as one of the antecedents is to one of the consequents, so are all the antecedents to all the consequents; [VII. 12]

therefore, as KH is to FH, so are EL, LK, KH to LF, FK, HF.

But KH is equal to CG, FH to A, and LF, FK, HF to D, BC, A;

therefore, as CG is to A, so is EH to D, BC, A.

Therefore, as the excess of the second is to the first, so is the excess of the last to all those before it.

Q. E. D.

This proposition is perhaps the most interesting in the arithmetical Books, since it gives a method, and a very elegant one, of *summing any series of terms in geometrical progression.*

Let $a_1, a_2, a_3, \ldots a_n, a_{n+1}$ be a series of terms in geometrical progression. Then Euclid's proposition proves that

$$(a_{n+1} - a_1) : (a_1 + a_2 + \ldots + a_n) = (a_2 - a_1) : a_1.$$

For clearness' sake we will on this occasion use the fractional notation of algebra to represent proportions.

Euclid's method then comes to this.

Since $$\frac{a_{n+1}}{a_n} = \frac{a_n}{a_{n-1}} = \ldots = \frac{a_2}{a_1},$$

we have, *separando*,

$$\frac{a_{n+1} - a_n}{a_n} = \frac{a_n - a_{n-1}}{a_{n-1}} = \ldots = \frac{a_3 - a_2}{a_2} = \frac{a_2 - a_1}{a_1},$$

whence, since, as one of the antecedents is to one of the consequents, so is the sum of all the antecedents to the sum of all the consequents, [VII. 12]

$$\frac{a_{n+1} - a_1}{a_n + a_{n-1} + \ldots + a_1} = \frac{a_2 - a_1}{a_1},$$

which gives $a_1 + a_2 + \ldots + a_n$, or S_n.

If, to compare the result with that arrived at in algebraical text-books, we write the series in the form

$$a, \quad ar, \quad ar^2, \ldots ar^{n-1} \quad (n \text{ terms}),$$

we have $$\frac{ar^n - a}{S_n} = \frac{ar - a}{a},$$

or $$S_n = \frac{a(r^n - 1)}{r - 1}.$$

PROPOSITION 36.

If as many numbers as we please beginning from an unit be set out continuously in double proportion, until the sum of all becomes prime, and if the sum multiplied into the last make some number, the product will be perfect.

For let as many numbers as we please, A, B, C, D, beginning from an unit be set out in double proportion, until the sum of all becomes prime,

let E be equal to the sum, and let E by multiplying D make FG;

I say that FG is perfect.

For, however many A, B, C, D are in multitude, let so many E, HK, L, M be taken in double proportion beginning from E;

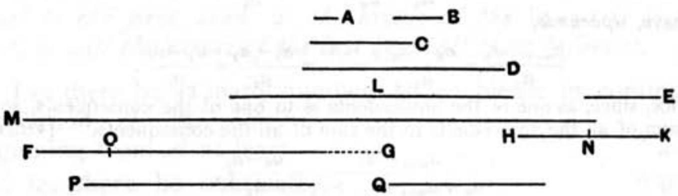
therefore, *ex aequali*, as A is to D, so is E to M. [VII. 14]

Therefore the product of E, D is equal to the product of A, M. [VII. 19]

And the product of E, D is FG;
therefore the product of A, M is also FG.

Therefore A by multiplying M has made FG;
therefore M measures FG according to the units in A.

And A is a dyad;
therefore FG is double of M.



But M, L, HK, E are continuously double of each other; therefore E, HK, L, M, FG are continuously proportional in double proportion.

Now let there be subtracted from the second HK and the last FG the numbers HN, FO, each equal to the first E; therefore, as the excess of the second is to the first, so is the excess of the last to all those before it. [IX. 35]

Therefore, as NK is to E, so is OG to M, L, KH, E.

And NK is equal to E;
therefore OG is also equal to M, L, HK, E.

But FO is also equal to E,
and E is equal to A, B, C, D and the unit.

Therefore the whole FG is equal to E, HK, L, M and A, B, C, D and the unit;
and it is measured by them.

I say also that FG will not be measured by any other number except A, B, C, D, E, HK, L, M and the unit.

For, if possible, let some number P measure FG,
and let P not be the same with any of the numbers A, B, C, D, E, HK, L, M.

And, as many times as P measures FG, so many units let there be in Q;
therefore Q by multiplying P has made FG.

But, further, E has also by multiplying D made FG;
therefore, as E is to Q, so is P to D. [VII. 19]

And, since A, B, C, D are continuously proportional beginning from an unit,
therefore D will not be measured by any other number except A, B, C. [IX. 13]

And, by hypothesis, P is not the same with any of the numbers A, B, C;
therefore P will not measure D.

But, as P is to D, so is E to Q;
therefore neither does E measure Q. [VII. Def. 20]

And E is prime;
and any prime number is prime to any number which it does not measure. [VII. 29]

Therefore E, Q are prime to one another.

But primes are also least, [VII. 21]
and the least numbers measure those which have the same ratio the same number of times, the antecedent the antecedent and the consequent the consequent; [VII. 20]
and, as E is to Q, so is P to D;
therefore E measures P the same number of times that Q measures D.

But D is not measured by any other number except A, B, C;
therefore Q is the same with one of the numbers A, B, C.

Let it be the same with B.

And, however many B, C, D are in multitude, let so many E, HK, L be taken beginning from E.

Now E, HK, L are in the same ratio with B, C, D;
therefore, *ex aequali*, as B is to D, so is E to L. [VII. 14]

Therefore the product of B, L is equal to the product of D, E. [VII. 19]

But the product of D, E is equal to the product of Q, P;
therefore the product of Q, P is also equal to the product of B, L.

Therefore, as Q is to B, so is L to P. [VII. 19]

And Q is the same with B;
therefore L is also the same with P:

which is impossible, for by hypothesis P is not the same with any of the numbers set out.

Therefore no number will measure FG except A, B, C, D, E, HK, L, M and the unit.

And FG was proved equal to A, B, C, D, E, HK, L, M and the unit;

and a perfect number is that which is equal to its own parts; [VII. Def. 22]

therefore FG is perfect.

Q. E. D.

If the sum of any number of terms of the series

$$1,\ 2,\ 2^2,\ \dots\ 2^{n-1}$$

be prime, and the said sum be multiplied by the last term, the product will be a "perfect" number, i.e. equal to the sum of all its factors.

Let $1 + 2 + 2^2 + \dots + 2^{n-1}$ $(= S_n)$ be prime;

then shall $S_n \cdot 2^{n-1}$ be "perfect."

Take $(n - 1)$ terms of the series

$$S_n,\ 2S_n,\ 2^2S_n,\ \dots\ 2^{n-2}S_n.$$

These are then terms proportional to the terms

$$2,\ 2^2,\ 2^3,\ \dots\ 2^{n-1}.$$

Therefore, *ex aequali*,

$$2 : 2^{n-1} = S_n : 2^{n-2} S_n, \qquad \text{[VII. 14]}$$

or

$$2 \cdot 2^{n-2}S_n = 2^{n-1} \cdot S_n. \qquad \text{[VII. 19]}$$

(This is of course obvious algebraically, but Euclid's notation requires him to prove it.)

Now, by IX. 35, we can sum the series $S_n + 2S_n + \dots + 2^{n-2}S_n$,

and $\quad (2S_n - S_n) : S_n = (2^{n-1} S_n - S_n) : (S_n + 2S_n + \dots + 2^{n-2}S_n)$.

Therefore $\quad S_n + 2S_n + 2^2S_n + \dots + 2^{n-2}S_n = 2^{n-1}S_n - S_n$,

or

$$2^{n-1}S_n = S_n + 2S_n + 2^2S_n + \dots + 2^{n-2}S_n + S_n$$
$$= S_n + 2S_n + \dots + 2^{n-2}S_n + (1 + 2 + 2^2 + \dots + 2^{n-1}),$$

and $2^{n-1} S_n$ is measured by every term of the right hand expression.

It is now necessary to prove that $2^{n-1}S_n$ cannot have any factor except those terms.

Suppose, if possible, that it has a factor x different from all of them, and let

$$2^{n-1} S_n = x \cdot m.$$

Therefore

$$S_n : m = x : 2^{n-1}. \qquad \text{[VII. 19]}$$

Now 2^{n-1} can only be measured by the preceding terms of the series $1,\ 2,\ 2^2, \dots 2^{n-1}$, [IX. 13]

and x is different from all of these;

therefore x does not measure 2^{n-1},

so that S_n does not measure m. [VII. Def. 20]

And S_n is prime; therefore it is prime to m. [VII. 29]

It follows [VII. 20, 21] that

$$m \text{ measures } 2^{n-1}.$$

Suppose that $m = 2^r$.

Now, *ex aequali*, $2^r : 2^{n-1} = S_n : 2^{n-r-1} S_n$.

Therefore $2^r . 2^{n-r-1} S_n = 2^{n-1} S_n$ [VII. 19]

$= x . m$, from above.

And $m = 2^r$;

therefore $x = 2^{n-r-1} S_n$, one of the terms of the series $S_n, 2S_n, 2^2 S_n, \dots 2^{n-2} S_n$: which contradicts the hypothesis.

There $2^{n-1} S_n$ has no factors except

$$S_n, 2S_n, 2^2 S_n, \dots 2^{n-2} S_n, 1, 2, 2^2, \dots 2^{n-1}.$$

Theon of Smyrna and Nicomachus both define a "perfect" number and give the law of its formation. Nicomachus gives four perfect numbers and no more, namely 6, 28, 496, 8128. He says they are formed in "ordered" fashion, there being one among the units (i.e. less than 10), one among the tens (less than 100), one among the hundreds (less than 1000) and one among the thousands (less than 10000); he adds that they terminate in 6 or 8 alternately. They do all terminate in 6 or 8, as can easily be proved by means of the formula $(2^n - 1) 2^{n-1}$ (cf. Loria, *Le scienze esatte nell' antica Grecia*, pp. 840—1), but not alternately, for the fifth and sixth perfect numbers both end in 6, and the seventh and eighth both end in 8. Iamblichus adds a tentative suggestion that perhaps there may be, in like manner, one perfect number among the "first myriads" (less than 10000^2), one among the "second myriads" (less than 10000^3), and so on. This is, as we shall see, incorrect.

It is natural that the subject of perfect numbers should, ever since Euclid's time, have had a fascination for mathematicians. Fermat (1601—1655), in a letter to Mersenne (*Œuvres de Fermat*, ed. Tannery and Henry, Vol. II., 1894, pp. 197—9), enunciated three propositions which much facilitate the investigation whether a given number of the form $2^n - 1$ is prime or not. If we write in one line the exponents 1, 2, 3, 4, etc. of the successive powers of 2 and underneath them respectively the numbers representing the corresponding powers of 2 diminished by 1, thus,

$$\begin{array}{cccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \dots n \\ 1 & 3 & 7 & 15 & 31 & 63 & 127 & 255 & 511 & 1023 & 2047 \dots 2^n - 1, \end{array}$$

the following relations are found to subsist between the numbers in the first line and those directly below them in the second line.

1. If the exponent is not a prime number, the corresponding number is not a prime number either (since $a^{pq} - 1$ is always divisible by $a^p - 1$ as well as by $a^q - 1$).

2. If the exponent is a prime number, the corresponding number diminished by 1 is divisible by twice the exponent. [$(2^n - 2)/2n = (2^{n-1} - 1)/n$; so that this is a special case of "Fermat's theorem" that, if p is a prime number and a is prime to p, then a^{p-1} is divisible by p.]

3. If the exponent n is a prime number, the corresponding number is only divisible by numbers of the form $(2mn + 1)$. If therefore the corresponding number in the second line has no factors of this form, it has no integral factor.

The first and third of these propositions are those which are specially useful for the purpose in question. As usual, Fermat does not give his proofs but merely adds: "Voilà trois fort belles propositions que j'ay trouvées et prouvées non sans peine. Je les puis appeller les fondements de l'invention des nombres parfaits."

I append a few details of discoveries of further perfect numbers after the first four. The next are as follows:

fifth, $2^{12}(2^{13}-1) = 33\ 550\ 336$
sixth, $2^{16}(2^{17}-1) = 8\ 589\ 869\ 056$
seventh, $2^{18}(2^{19}-1) = 137\ 438\ 691\ 328$
eighth, $2^{30}(2^{31}-1) = 2\ 305\ 843\ 008\ 139\ 952\ 128$
ninth, $2^{60}(2^{61}-1) = 2\ 658\ 455\ 991\ 569\ 831\ 744\ 654\ 692\ 615\ 953\ 842\ 176$
tenth, $2^{88}(2^{89}-1)$.

It has further been proved that $2^{107}-1$ is prime, and so is $2^{127}-1$. Hence $2^{106}(2^{107}-1)$ and $2^{126}(2^{127}-1)$ are two more perfect numbers.

The fifth perfect number may have been known to Iamblichus, though he does not give it; it was however known, with all its factors, in the fifteenth century, as appears from a tract written in German which was discovered by Curtze (Cod. lat. Monac. 14908). The first eight perfect numbers were calculated by Jean Prestet (d. 1670). Fermat had stated, and Euler proved, that $2^{31}-1$ is prime. The ninth perfect number was found by P. Seelhoff (*Zeitschrift für Math. u. Physik*, XXXI., 1886, pp. 174—8) and verified by E. Lucas (*Mathésis*, VII., 1887, pp. 45—6). The tenth was discovered by R. E. Powers (see *Bulletin of the American Mathematical Society*, XVIII., 1912, p. 162). $2^{107}-1$ was proved to be prime by E. Fauquembergue and R. E. Powers (1914), while Fauquembergue proved that $2^{127}-1$ is prime.

There have been attempts, so far unsuccessful, to solve the question whether there exist other "perfect numbers" than those of Euclid, and, in particular, perfect numbers which are *odd*. (Cf. several notes by Sylvester in *Comptes rendus*, CVI., 1888; Catalan, "Mélanges mathématiques" in *Mém. de la Soc. de Liége*, 2e Série, XV., 1888, pp. 205—7; C. Servais in *Mathésis*, VII., pp. 228—30 and VIII., pp. 92—93, 135; E. Cesàro in *Mathésis*, VII., pp. 245—6; E. Lucas in *Mathésis*, X., pp. 74—6).

For the detailed history of the whole subject see L. E. Dickson, *History of the Theory of Numbers*, Vol. I., 1919, pp. iii—iv, 3—33.