Recursion and Induction

Notes on Mathematical Logic and Model Theory

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Preface

> The beginner *has in his head a definition of the science*; a childish definition perhaps, but still a definition; of the science's subject-matter he has *no definition at all.*
>
> Only the hope of a definition. 'I don't know what life, is, but I hope I shall when I have studied physiology for long enough.'
>
> 'That is true for a beginner in physiology; but for a master in physiology the reverse is true; a master in physiology has found out all that it can tell him and knows what life is. A beginner in physiology does not; for him physiology is definable and life as yet, except in the language of hope, indefinable.'
>
> A man ceases to be a beginner in any given science and becomes a master in that science when he has learned that *this expected reversal is never going to happen* and that he is going to be a beginner all his life.
>
> —R. G. Collingwood, *The New Leviathan* [**9**, 1.43–46]

These notes are for use in a course called Introduction to Mathematical Logic and Model Theory, Math 406, given at METU in the fall of 2008. The notes are based on the notes I prepared while teaching the course in the fall of 2004. But I have made many changes and additions.

The title of these notes refers to the methods of *recursion* and *induction.* I have become increasingly aware of how these two methods are confused. In these notes, recursion is a method of *definition;* induction is a method of *proof.* If a set is defined by recursion, then properties of elements of the set can be proved by induction. However, it does not then necessarily follow that functions *on* the set can be defined by recursion. Logic provides examples of this phenomenon and a way to understand it.

These notes are intended as a supplement to the classroom experience, and not for independent study. I say this because the notes may not give full explanations of some matters; they may give too much explanation of other matters; and they may have mistakes and other features to be changed during the course. Various examples and topics are left as exercises: investigation of some of these will depend on the interest of the student.

I first learned logic from David Kueker and Chris Laskowski, and from the notes that I took in their courses and that I still consult today. Another influence on these notes is the book [**8**] of Alonzo Church: as a student, I obtained a leftover display copy of this at a meeting of the Association for Symbolic Logic. References for current model theory include Hodges [**16**], Marker [**22**], and Rothmaler [**29**]; but this is not a complete list of the books consulted in the preparation of these notes. Shoenfield [**31**, p. iv] is a good source—though dated—for mathematical logic in general. Concerning his practice of attribution, he writes

> I have made no attempt to credit each result to its author; the names attached to the principal theorems are there simply to give the reader some idea of the people who have created the subject. I have also omitted all bibliographical references.

My practice is not so extreme. Most of what is in these notes has been worked out only since the late 1800s, so it is possible to track down the original sources. I have done this in a few cases. For sources in the other cases, especially in model theory, Hodges [**16**] would be the place to look.

Conventions

The lemma called Lemma 5.3.2, for example, is the second lemma in §5.3 (namely, Section Three of Chapter Five). Displayed expressions that will be referred to later are labelled from the sequence

$$(*) \quad (\dagger) \quad (\ddagger) \quad (\S) \quad (\P) \quad (\|) \quad (**) \quad (\dagger\dagger) \quad (\ddagger\ddagger)$$

But the labels repeat. Hence a reference to $(*)$ is a reference to the *last* displayed expression labelled as $(*)$.

Proofs begin with the word *Proof* and end with a box $\Box$. If there is no proof given, then supplying it is an exercise. Other exercises are indicated in the text; these are repeated, and more are added, at the ends of chapters.

I also put technical terms in **boldface** when they are being defined (perhaps only implicitly). If they are only being emphasized for some other reason, then they may be *slanted.* All such terms are listed in the index at the back. Throughout the text, ordinary *italics* and 'quotation marks' are used for the usual sorts of reasons.

CHAPTER 1

Introduction

1.1. Building blocks

An **ordered pair** is defined by the identity

$$(x, y) = \{\{x\}, \{x, y\}\}.$$

The sole purpose of the definition is to ensure that

$$(x, y) = (a, b) \iff x = a \ \& \ y = b.$$

(The sign $\iff$ is just an abbreviation of the English *if and only if*.) The **Cartesian product** of sets A and B is the set

$$\{(x, y) \colon x \in A \ \& \ y \in B\},$$

which is denoted by

$$A \times B.$$

A **relation** from A to B is just a subset of $A \times B$. If R is such a relation, and $(a, b) \in R$, then we may also write

$$a \mathrel{R} b.$$

The **domain** of R is given by

$$\operatorname{dom}(R) = \{x \colon \exists y \ x \mathrel{R} y\};$$

that is, the domain of R is the set of a for which there is some b such that $a \mathrel{R} b$.

A relation R from A to B is a **function** from A to B if, for every element a of A, there is a unique element b of B such that $a \mathrel{R} b$. If f is a function from A to B, then we can express this by writing

$$f \colon A \longrightarrow B.$$

The set B is the **codomain** of f, but (unlike the domain) it is not determined by f alone. If $a \mathrel{f} b$, then we usually write

$$f(a) = b$$

instead; also, b is the **image** of a under f. The function f itself can also be written as

$$x \longmapsto f(x).$$

The function f is **injective** if for each b in B there is at *most* one element a of A such that $f(a) = b$; **surjective,** if for each b in B there is at *least* one such element a of A; **bijective,** if both injective and surjective. The set $\{y \colon \exists x \ f(x) = y\}$ or $\{f(x) \colon x \in A\}$ is the **image** or **range** of f; so f is surjective if and only if this image is B. If $C \subseteq A$, then the **restriction** of f to C is given by

$$f \restriction C = f \cap (C \times B).$$

A relation from A to itself is a **binary relation** on A. One such relation is the **diagonal,** given by

$$\Delta_A = \{(x,y)\colon x = y \ \&\ x \in A\}.$$

This is a function from A to itself, namely the **identity;** considered as such, it may be denoted by

$$\mathrm{id}_A\,.$$

If $R \subseteq A \times B$, then the **converse** of R is given by

$$\breve{R} = \{(y,x)\colon x\ R\ y\}.$$

If also $S \subseteq B \times C$, then the **composite** of R and S is the relation from A to C given by

$$R/S = \{(x,z)\colon \exists y\ (x\ R\ y \ \&\ y\ S\ z)\}.$$

The point of these derived relations is to allow some clever definitions of certain kinds of relations. So, R is a function from A to B if and only if $\Delta_A \subseteq R/\breve{R}$ and $\breve{R}/R \subseteq \Delta_B$. Assuming $f\colon A \to B$, we have that f is injective if and only if $f/\breve{f} \subseteq \Delta_A$, and surjective if and only if $\Delta_B \subseteq \breve{f}/f$. If also $g\colon B \to C$, then

$$g \circ f = f/g.$$

A binary relation R on A is **reflexive,** if $\Delta_A \subseteq R$; **irreflexive,** if $R \cap \Delta_A = \varnothing$; **symmetric,** if $R = \breve{R}$; **antisymmetric,** if $R \cap \breve{R} \subseteq \Delta_A$; **transitive,** if $R/R \subseteq R$. A relation is an **equivalence-relation,** or just an **equivalence,** if it is reflexive, symmetric, and transitive; an **ordering,** if antisymmetric, transitive, and either reflexive or irreflexive. An irreflexive ordering is also called **strict.** An ordering R of A is **total** if $R \cup \breve{R} \cup \Delta_A = A \times A$; otherwise the ordering is **partial.**[1] If R is an ordering of A, then the pair (A,R) is an **order.**

A subset of A is a **singulary**[2] **relation** on A. A **ternary relation** on A is a subset of $A \times A \times A$, and so forth.

A **singulary operation** on A is a function from A to itself; a **binary operation** on A is a function from $A \times A$ to A. Taking Cartesian products is itself a binary operation,

$$(A,B) \mapsto A \times B,$$

on the *class* of sets; taking **power-sets,** that is, sets of subsets, is a singulary operation,

$$A \mapsto \mathcal{P}(A),$$

on the class of sets. An element of a set can be considered as a **nullary operation** on the set.

[1] For some writers, 'partial ordering' means ordering, and 'ordering' means total ordering.

[2] The word **unary** is often used instead of *singulary.* Following Quine, Church [**8**, § 02, p. 12, n. 29] suggests *singulary* as a more etymologically correct word than *unary.* Indeed, whereas the first five Latin cardinal numbers are UN-, DU-, TRI-, QUATTUOR, QUINQUE, the first five Latin *distributive* numbers—corresponding to the Turkish birer, ikişer, üçer, dörder, beşer [**25**]—are SINGUL-, BIN-, TERN-, QUATERN-, QUIN-. It is the latter sequence that gives us *binary* and *ternary*—also *quaternary* and *quinary*, if these are desired. So *singulary* appears to be a better word than *unary.* In fact, *singulary* does not appear in the original *Oxford English Dictionary* [**24**]. The word *unary does* appear in this dictionary, but it is considered obsolete: only one use of the word, from 1576, was discovered in English literature. There, *unary* meant *unit*, although the word *unit* was not actually invented until 1570, when it was introduced by [John] Dee to correspond to the Greek $\mu o \nu \alpha \delta$-.

1.2. Model theory

> To think that physics or chemistry ought to be defined in terms of matter or physiology in terms of life is more than an egregious blunder; it is a threat to the existence of science.
>
> It implies that people know what matter is without studying physics or chemistry, and what life is without studying physiology.
>
> It implies that this non-scientific and pre-scientific knowledge concerning the nature of matter or life is perfect and final, so far as it goes, and can never be corrected by anything science can do.
>
> It implies that, if anything scientists imagine themselves to have discovered about matter or life or what not is inconsistent with anything contained or implied in this non-scientific and pre-scientific knowledge, the scientists have made a mistake.
>
> It implies that, if they have made the mistake by using (for example) experimental methods, it is experimental methods that are at fault and must be abandoned.
>
> It implies that, if they have made the mistake by arguing logically, it is logic that is at fault and must be abandoned.
>
> It implies that any scientists who will not yield to persuasion and confess the supremacy of non-scientific or pre-scientific knowledge over all possible scientific inquiry must be made to yield by any means that can be devised.
>
> At one blow, by enunciating the apparently harmless proposition that physics or chemistry is the science of matter, physiology the science of life, or the like, we have evoked the whole apparatus of *scientific persecution*; I mean the persecution of scientists for daring to be scientists.
>
> In whose interest is such a persecution carried on? Who stands to gain by it? The nominal beneficiary differs from time to time: sometimes it is religion, sometimes statecraft, and so on. None of these has ever in fact gained a ha'porth of advantage. The actual beneficiary has always been *obsolete science.*
>
> —R. G. Collingwood, *The New Leviathan* [**9**, I.5–58]

Model theory is whatever is taught in courses and books that have model theory in their titles. Different writers will give different definitions of what model theory is. In my view, model theory is a kind of mathematics done self-consciously. It is mathematics done while paying attention to what it *means* to do mathematics. In particular, model theory pays attention to the *language* of mathematics. For a simple example, the sets commonly denoted by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ can all be talked about in a language whose special symbols include $+$ and $\times$: this is a model-theoretic observation.

Model theory is the study of *structures quâ models* of *theories.* A brief elucidation of the technical terms in this definition might run as follows.

Examples of *structures* include the sets like $\mathbb{N}$ that have just been mentioned, when these are considered as being equipped with the named operations of addition and multiplication. An order in the sense of §1.1, or a group, is a structure. A set by itself is the simplest kind of structure. In general, a **structure** is a set equipped with some (or no) operations and relations on it. The set itself may be called the **universe** of the structure. A convention that I like to follow is that, if the universe is denoted by a plain letter, as A

or B, then the whole structure is denoted by the Fraktur form of that letter, as $\mathfrak{A}$ or $\mathfrak{B}$. (See Appendix A.) But when the universe is a standard set that is already denoted by a fancy letter, as $\mathbb{N}$, then this letter may also be understood to denote the whole structure with that universe. We may refer to an element of a structure when we mean an element of its universe.

The word *quâ* might be rendered literally in Turkish as *ondan* or *neden,* though Redhouse [**11**] suggests *sıfatıyle* and *niteliğinde.* It is a Latin relative pronoun in the ablative case, used in technical English to mean *in the capacity of.* It is perhaps originally a translation of the Greek ᾗ, a relative pronoun in the dative case, used for example by Aristotle [**2**, IV, 1003a21] in referring to τὸ ὂν ᾗ ὄν, *being as such,* as the subject of the work now known as the *Metaphysics.* (On the Greek alphabet, see Appendix B.)

The special symbols for the operations and relations of a structure (such as $+$ and $\times$ for $\mathbb{N}$ or $\mathbb{Q}$) constitute its **signature.** *Sentences* in this signature are either true or false in the structure. A **model** of a set of sentences is a structure in which all of the sentences are true. A set of sentences is **theory** if it contains all of the sentences that are true in all of its models. The theory *of* a structure is the set of sentences that are true in the structure. This theory may have models that are fundamentally different from the original structure: this is one feature that makes model theory interesting.

1.3. Use and mention

Of the following three sentences, the first two **use** the word ice, while the last two **mention** this word.

(1) Ice is frozen water.
(2) Ice has one syllable.
(3) The English word for frozen water has one syllable.

Note what happens when the sentences are translated into Turkish:

(1) *Buz donmuş sudur.*
(2) Ice*'in bir hecesi vardır.*
(3) *Donmuş suyun İngilizcesinin bir hecesi vardır.*

In sentence (2), the word ice is used to mention itself. This self-referential use of a word may be shown typographically, by using quotation-marks around the word, or setting it in a different font:

'Ice' has one syllable.
Ice has one syllable.

But there need be no typographical distinction at all, as long as context makes the intended use of a word clear.

The distinction between use and mention of an expression can be seen in mathematics, as in

(1) $2 + 2 = 4$.
(2) The sign $+$ denotes addition.
(3) A sign resembling a Greek cross denotes addition.

Occasionally a word is not precise enough to make its use clear. For example, David wrote *A Treatise of Human Nature,* and David is writing the present book: but this is confusing or misleading. David Hume wrote the former book [**18**], which was published in 1739–40, while David Pierce is writing the latter. Thus we can attach tags to the name David to show which David is meant in each case.

The same occasional need for greater precision arises in mathematics. For example, a group-homomorphism is a function f from one group to another for which the equation

$$f(x \cdot y) = f(x) \cdot f(y) \tag{$*$}$$

is an identity. Here it must be understood that the dot on the left-hand side of the equation refers to multiplication in the domain of f; on the right, the co-domain. If $f\colon G \to H$, then by adding labels to the dots, we can write ($*$) more precisely as

$$f(x \cdot^G y) = f(x) \cdot^H f(y).$$

1.4. The natural numbers

The **natural numbers** compose a set with the following five properties.

(1) There is an **initial element.**
(2) Every element has a unique **successor.**
(3) A subset is the whole set if it contains the initial element and contains the successor of each of its own elements.
(4) The initial element is not the successor of any element.
(5) Elements with the same successor are the same.

These properties were identified by Richard Dedekind [**10**]; they were then written out in a new logical notation by Giuseppe Peano [**26**], and they have come to be called the **Peano axioms.**

Let us denote the set of natural numbers by $\mathbb{N}$; its initial element, by 1; and the successor of an element k, by $\mathrm{s}(k)$ or by k^{s}, according to convenience. Then we have a structure, which we may denote by

$$(\mathbb{N}, 1, \mathrm{s}), \tag{$*$}$$

whose universe is $\mathbb{N}$, and whose signature is

$$\{1, \mathrm{s}\}. \tag{$\dagger$}$$

This is just what properties (1) and (2) give us. Note that, in ($*$), the symbols 1 and s refer to an element and a singulary operation respectively, while in (†), they refer to themselves. If we do not like this ambiguity, then we may rewrite ($*$) as

$$(\mathbb{N}, 1^{\mathbb{N}}, \mathrm{s}^{\mathbb{N}}).$$

As noted in § 1.2, the single letter $\mathbb{N}$ may also be understood to denote this whole structure. Property (3) is that this structure admits **(proof by) induction.** The remaining two properties can be written out more formally thus:

(4) $\forall x\ 1 \neq x^{\mathrm{s}}$.
(5) $\forall x\ \forall y\ (x^{\mathrm{s}} = y^{\mathrm{s}} \Rightarrow x = y)$.

This last property is just that the successor-operation is injective.

There are stuctures in the signature $\{1, \mathrm{s}\}$ that admit proof by induction, but do not have properties (4) and (5). For example, suppose A is a three-element set, as $\{c, d, e\}$. We obtain a structure $\mathfrak{A}$ in the signature $\{1, \mathrm{s}\}$ when we define $1^{\mathfrak{A}}$ as c and define $\mathrm{s}^{\mathfrak{A}}$ by the following table.

x	c	d	e
x^{s}	d	e	c

Then $\mathfrak{A}$ admits proof by induction. Indeed, every subset of A that contains c and the successors of its elements must contain d and e, so it must be all of A. But the initial element in $\mathfrak{A}$ is a successor. We can define a new structure $\mathfrak{B}$ with the same universe $\{c, d, e\}$, and the same initial element c, but where $\mathrm{s}^{\mathfrak{B}}$ is as follows.

x	c	d	e
x^{s}	d	e	d

Now $\mathfrak{B}$ admits proof by induction, and moreover, the initial element is not a successor; but two distinct elements have the same successor.

Let us refer to all structures in the signature $\{1, \mathrm{s}\}$ as **iterative structures;**[3] and let us refer to iterative structures that admit proof by induction as **inductive structures.** So $\mathbb{N}$ is an inductive structure, as are the other two structures just defined. A basic consequence of induction is the following.

THEOREM 1.4.1. *Every element of an inductive structure is either the initial element or a successor.*

PROOF. Suppose $\mathfrak{A}$ is an inductive structure. Let M be the set of elements of A that are either the initial element or successors. Then the initial element is in M, and so is the successor of every element of M, just because it is a successor. By induction, $M = A$. $\square$

A **homomorphism** between two iterative structures is a function from (the universe of) one to (the universe of) the other that takes the initial element to the initial element and takes the successor of every element to the successor of its image. So, if $\mathfrak{A}$ and $\mathfrak{B}$ are arbitrary iterative structures, and $f\colon A \to B$, then f is a homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$ just in case

(1) $f(1^{\mathfrak{A}}) = 1^{\mathfrak{B}}$;
(2) $f(\mathrm{s}^{\mathfrak{A}}(c)) = \mathrm{s}^{\mathfrak{B}}(f(c))$ for all c in A, that is, $f \circ \mathrm{s}^{\mathfrak{A}} = \mathrm{s}^{\mathfrak{B}} \circ f$.

An iterative structure $\mathfrak{A}$ admits **recursion** if, for every iterative structure $\mathfrak{B}$, there is a *unique* homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$. In this case, that unique homomorphism is said to be **recursively defined.**

THEOREM 1.4.2 (Recursion). *Every structure that satisfies the Peano axioms admits recursion.*

PROOF. Suppose $\mathfrak{A}$ meets the given five conditions, and $\mathfrak{B}$ is another iterative structure. We show that there is a unique homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$.

Assuming existence for the moment, we can prove uniqueness by induction. Indeed, suppose f and g are homomorphisms from $\mathfrak{A}$ to $\mathfrak{B}$. Let M be the subset of A comprising those c such that $f(c) = g(c)$. Because f and g are homomorphisms, we have

$$f(1^{\mathfrak{A}}) = 1^{\mathfrak{B}} = g(1^{\mathfrak{A}}),$$

so $1^{\mathfrak{A}} \in M$. Suppose $d \in M$, so $f(d) = g(d)$. Again since f and g are homomorphisms, we have

$$f(\mathrm{s}^{\mathfrak{A}}(d)) = \mathrm{s}^{\mathfrak{B}}(f(d)) = \mathrm{s}^{\mathfrak{B}}(g(d)) = g(\mathrm{s}^{\mathfrak{A}}(d)),$$

so $\mathrm{s}^{\mathfrak{A}}(d) \in M$. By induction, $M = A$, so $f = g$.

It remains to show that such a homomorphism f exists at all. We want to say that f is the set

$$\{(1^{\mathfrak{A}}, 1^{\mathfrak{B}}), (\mathrm{s}^{\mathfrak{A}}(1^{\mathfrak{A}}), (\mathrm{s}^{\mathfrak{B}}(1^{\mathfrak{B}})), (\mathrm{s}^{\mathfrak{A}}(\mathrm{s}^{\mathfrak{A}}(1^{\mathfrak{A}})), \mathrm{s}^{\mathfrak{B}}((\mathrm{s}^{\mathfrak{B}}(1^{\mathfrak{B}}))), \dots\}.$$

[3]This is my term, for want of a better; another possibility might be *discrete dynamical systems.*

We can write this set as $\{(1,1),(1^{\mathrm{s}},1^{\mathrm{s}}),(1^{\mathrm{ss}},1^{\mathrm{ss}}),\dots\}$, as long as we understand that the left entry in each ordered pair is in $\mathfrak{A}$; and the right, in $\mathfrak{B}$. In any case, we have to give a valid definition of this set, to ensure that it exists. One way to do this is to build up f as the union of the sets

$$\begin{array}{l}\{(1^{\mathfrak{A}},1^{\mathfrak{B}})\},\\ \{(1^{\mathfrak{A}},1^{\mathfrak{B}}),(\mathrm{s}^{\mathfrak{A}}(1^{\mathfrak{A}}),(\mathrm{s}^{\mathfrak{B}}(1^{\mathfrak{B}}))\},\\ \{(1^{\mathfrak{A}},1^{\mathfrak{B}}),(\mathrm{s}^{\mathfrak{A}}(1^{\mathfrak{A}}),(\mathrm{s}^{\mathfrak{B}}(1^{\mathfrak{B}})),(\mathrm{s}^{\mathfrak{A}}(\mathrm{s}^{\mathfrak{A}}(1^{\mathfrak{A}})),\mathrm{s}^{\mathfrak{B}}((\mathrm{s}^{\mathfrak{B}}(1^{\mathfrak{B}})))\},\\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\end{array}\tag{‡}$$

That is, if $\mathbf{C}$ is the set of all sets listed in (‡), then we let $f=\bigcup\mathbf{C}$. But we still have to give a valid criterion for membership in $\mathbf{C}$ (and then prove f is a homomorphism).

To be precise, we let $\mathbf{C}$ comprise those subsets D of $A\times B$ such that, if $(a,b)\in D$, then either

$$(a,b)=(1^{\mathfrak{A}},1^{\mathfrak{B}})=(1,1),$$

or else

$$(a,b)=(\mathrm{s}^{\mathfrak{A}}(c),\mathrm{s}^{\mathfrak{B}}(d))=(c^{\mathrm{s}},d^{\mathrm{s}})$$

for some c and d such that $(c,d)\in D$. Now $\mathbf{C}$ is well defined. Let $R=\bigcup\mathbf{C}$. Then R is a well-defined relation from A to B. Moreover, since $\{(1,1)\}\in\mathbf{C}$, we have $1\mathrel{R}1$; and if $c\mathrel{R}d$, so that $(c,d)\in D$ for some D in $\mathbf{C}$, then $D\cup\{(c^{\mathrm{s}},d^{\mathrm{s}})\}\in\mathbf{C}$, so $c^{\mathrm{s}}\mathrel{R}d^{\mathrm{s}}$. Thus R is indeed a homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$, provided it is a *function* from A to B.

We can already conclude that, for every a in A, there is b in B such that $a\mathrel{R}b$. It remains to prove, assuming $a\mathrel{R}b$, that this b is unique. For this, we shall use the additional properties of $\mathfrak{A}$. Suppose $1\mathrel{R}b$. Then $(1,b)\in D$ for some D in $\mathbf{C}$. In $\mathfrak{A}$, we assume that 1 is not a successor. Therefore, by definition of $\mathbf{C}$, we know that $b=1$.

Now suppose that, for some c in A, there is a unique d in B such that $c\mathrel{R}d$. We know $c^{\mathrm{s}}\mathrel{R}d^{\mathrm{s}}$. Suppose $c^{\mathrm{s}}\mathrel{R}e$. Then $(c^{\mathrm{s}},e)\in D$ for some D in $\mathbf{C}$. Since 1 is not a successor in $\mathfrak{A}$, we must have $(k,\ell)\in D$ for some k and ℓ such that $(k^{\mathrm{s}},\ell^{\mathrm{s}})=(c^{\mathrm{s}},e)$. Since the successor-operation is injective, we have $k=c$. Since $k\mathrel{R}\ell$, this means $c\mathrel{R}\ell$. By uniqueness of d, since $c\mathrel{R}d$, we conclude $\ell=d$, so $e=\ell^{\mathrm{s}}=d^{\mathrm{s}}$.

By induction, R is a function from A to B. This completes the proof. □

The Recursion Theorem allows us to define the usual arithmetic operations of addition and multiplication and exponentiation on $\mathbb{N}$. In particular, we can write s as $x\mapsto x+1$. In fact, addition and multiplication can be defined in any *inductive* structure; but not exponentiation. See Appendix C.

A modification of the Recursion Theorem is

Corollary. *Suppose A is a set with an element b, and $F\colon\mathbb{N}\times A\to A$. Then there is a unique function G from $\mathbb{N}$ to A such that*

(1) $G(1)=b$, and
(2) $G(n+1)=F(n,G(n))$ for all n in $\mathbb{N}$.

Proof. Let $f\colon\mathbb{N}\times A\to\mathbb{N}\times A$, where $f(n,x)=(n+1,F(n,x))$. By recursion, there is a unique function g from $\mathbb{N}$ to $\mathbb{N}\times A$ such that $g(1)=(1,b)$ and $g(n+1)=f(g(n))$. By induction, the first entry in $g(n)$ is always n. The desired function G is given by $g(n)=(n,G(n))$. Indeed, we now have $G(1)=b$; also, $g(n+1)=f(n,G(n))=(n+1,F(n,G(n)))$, so $G(n+1)=F(n,G(n))$. By induction, G is unique. □

This allows taking factorials: again, see Appendix C.

An **isomorphism** is a homomorphism whose underlying function is a bijection whose inverse is also a homomorphism. To the Recursion Theorem then, we have the following converse.

THEOREM 1.4.3. *Every iterative structure that admits recursion is isomorphic to* $\mathbb{N}$ *and therefore satisfies the Peano axioms; in particular, it admits induction.*

PROOF. Suppose $\mathfrak{A}$ is an iterative structure that admits recursion. Then there are homomorphisms f from $\mathfrak{A}$ to $\mathbb{N}$ and g from $\mathbb{N}$ to $\mathfrak{A}$. Then $f \circ g$ is a homomorphism from $\mathbb{N}$ to itself. But the identity on $\mathbb{N}$ is also a homomorphism from $\mathbb{N}$ to itself; therefore $f \circ g$ must be the identity. For the same reason, $g \circ f$ is the identity on A. Thus f is invertible as a homomorphism, so it is an isomorphism. $\square$

The sets of *predecessors* of natural numbers are defined recursively. That is, we have $x \mapsto \operatorname{pred}(x) \colon \mathbb{N} \to \mathcal{P}(\mathbb{N})$, where

$$\begin{aligned} \operatorname{pred}(1) &= \varnothing; \\ \operatorname{pred}(n^{\mathrm{s}}) &= \operatorname{pred}(n) \cup \{n\}. \end{aligned}$$

The elements of $\operatorname{pred}(n)$ are the **predecessors** of n. We can define the binary relation $<$ on $\mathbb{N}$ by

$$x < y \iff x \in \operatorname{pred}(y). \tag{§}$$

Conversely, if $<$ is a binary relation on a set A, then (§) defines a predecessor-function from A to $\mathcal{P}(A)$. Then the **relational structure** $(A, <)$ admits **(proof by) induction** if, for every proper subset B of A, there is an element c of A that B does not contain, although B contains all predecessors of c. This means a subset C of A can be proved equal to A, provided that, from the **inductive hypothesis** that $\operatorname{pred}(d) \subseteq C$, it can be proved that $d \in C$.

Note well that we now have *two* kinds of induction. Context must be relied on to show which kind is meant.

THEOREM 1.4.4 (Induction). $(\mathbb{N}, <)$ *admits proof by induction.*

PROOF. Suppose B is a subset of $\mathbb{N}$ such that, if $\operatorname{pred}(d) \subseteq B$, then $d \in B$. We shall show $B = \mathbb{N}$. Let C comprise those elements of $\mathbb{N}$ whose predecessors belong to B. As 1 has no predecessors, they belong to B, so $1 \in C$. Suppose $n \in C$. Then all predecessors of n belong to B, so by assumption, $n \in B$. Thus, all predecessors of n^{s} belong to B, so $n^{\mathrm{s}} \in C$. By induction, $C = \mathbb{N}$. In particular, for all n in $\mathbb{N}$, we have $n^{\mathrm{s}} \in C$, so n (being a predecessor of n^{s}) belongs to B. Thus $B = \mathbb{N}$. $\square$

There are relational structures $(A, <)$ that admit induction, although $<$ is not transitive (exercise).

THEOREM 1.4.5. *The relation* $<$ *on* $\mathbb{N}$ *is transitive.*

PROOF. We show

$$x \in \operatorname{pred}(n) \implies \operatorname{pred}(x) \subseteq \operatorname{pred}(n)$$

(where $\implies$ stands for the English *implies*). The claim is vacuously true when $n = 1$, since $\operatorname{pred}(1) = \varnothing$. Suppose the claim is true when $n = m$. If $x \in \operatorname{pred}(m^{\mathrm{s}})$, then either $x \in \operatorname{pred}(m)$, or else $x = m$. In the former case, by inductive hypothesis, we have $\operatorname{pred}(x) \subseteq \operatorname{pred}(m)$; in the latter case, $\operatorname{pred}(x) = \operatorname{pred}(m)$. In either case, $\operatorname{pred}(x) \subseteq \operatorname{pred}(m) \subseteq \operatorname{pred}(m^{\mathrm{s}})$. By induction, we are done. $\square$

LEMMA 1.4.1. *Every transitive relational structure that admits induction is a strict order.* □

In particular, $(\mathbb{N}, <)$ is a strict order. But the ordering guaranteed by the lemma need not be total (exercise).

LEMMA 1.4.2. *In* $\mathbb{N}$, $1 \leqslant n$. □

LEMMA 1.4.3. *In* $\mathbb{N}$, *if* $m < n$, *then* $m^{\mathrm{s}} \leqslant n$.

PROOF. The claim is trivially true when $n = 1$. Suppose it is true when $n = k$. Say $m < k^{\mathrm{s}}$, that is, $m \in \operatorname{pred}(k) \cup \{k\}$. If $m \in \operatorname{pred}(k)$, then $m < k$, so $m^{\mathrm{s}} \leqslant k$ by inductive hypothesis, hence $m^{\mathrm{s}} \leqslant k^{\mathrm{s}}$. If $m = k$, then $m^{\mathrm{s}} = k^{\mathrm{s}}$, so $m^{\mathrm{s}} \leqslant k^{\mathrm{s}}$. □

THEOREM 1.4.6. $(\mathbb{N}, <)$ *is a strict total order.*

PROOF. We show
$$m \nleqslant n \implies n < m.$$
The claim is trivially true when $m = 1$, by Lemma 1.4.2. Suppose it is true when $m = k$. Say $k^{\mathrm{s}} \nleqslant n$. Then $k \nless n$ by the last lemma. If $k \neq n$, then $k \nleqslant n$, so $n < k < k^{\mathrm{s}}$ by inductive hypothesis. If $k = n$, then $n = k < k^{\mathrm{s}}$. □

A relational structure $(A, <)$ admits **recursion** if, for every set B and function f from $\mathcal{P}(B)$ into B, there is a unique function g from A to B such that
$$g(c) = f(g[\operatorname{pred}(c)]) \tag{¶}$$
for all c in A. Here $g[X]$ means $\{g(x) \colon x \in X\}$.

THEOREM 1.4.7. *A strict order that admits induction admits recursion.*

PROOF. Suppose $(A, <)$ is a strict order that admits induction. Let B be a set, and $f \colon \mathcal{P}(B) \to B$. Suppose there are functions h and h' from A to B such that (¶) holds for all c in A when g is h or h'. If h and h' agree on $\operatorname{pred}(d)$, then
$$h(d) = f(h[\operatorname{pred}(d)]) = f(h'[\operatorname{pred}(d)]) = h'(d),$$
so h and h' agree at d. By induction, $h = h'$.

It remains to show that such a function h exists at all. Let $\mathbf{C}$ comprise the relations R from A to B such that, if $a \mathrel{R} b$, then

(1) $\operatorname{pred}(a) \subseteq \operatorname{dom}(R)$;
(2) $b = f(\{y \colon \exists x\ (x < a\ \&\ x \mathrel{R} y)\})$.

Let $S = \bigcup \mathbf{C}$. We shall show that S is the desired function h. Let
$$S_a = S \cap (\operatorname{pred}(a) \times B).$$
Our inductive hypothesis is that S_a is a function h_a from $\operatorname{pred}(a)$ to B such that, when $c < a$, then
$$\begin{aligned} h_a(c) &= f(h_a[\operatorname{pred}(c)]) \\ &= f(\{y \colon \exists x\ (x < c\ \&\ h_a(x) = y)\}) \\ &= f(\{y \colon \exists x\ (x < c\ \&\ x \mathrel{S_a} y\}). \end{aligned}$$
By transitivity of $<$, if $c < a$, then $\operatorname{pred}(c) \subseteq \operatorname{pred}(a)$, so $\operatorname{pred}(c) \subseteq \operatorname{dom}(S_a)$. Hence $S_a \in \mathbf{C}$. Letting $b = f(h_a[\operatorname{pred}(a)])$, we have also $S_a \cup \{(a, b)\} \in \mathbf{C}$. Therefore $a \mathrel{S} b$.

Conversely, suppose $a\ S\ b'$. Then $a\ R\ b'$ for some R in $\mathbf{C}$. But then $\operatorname{pred}(a) \subseteq \operatorname{dom}(R)$, and

$$R \cap (\operatorname{pred}(a) \times B) \subseteq S \cap (\operatorname{pred}(a) \times B) = S_a,$$

so $R \cap (\operatorname{pred}(a) \times B) = S_a$. Therefore $b' = b$. This completes the induction and the proof. □

Hence $(\mathbb{N}, <)$ admits recursion.

A strict order $(A, <)$ is **well-founded** if every non-empty subset B of A has a **minimal element,** that is, an element c than which no element of B is less (this means we never have $d < c$ if $d \in B$).

THEOREM 1.4.8. *Strict orders that admit recursion are well-founded.*

PROOF. Suppose $(A, <)$ is a strict order that is not well-founded. Then A has a nonempty subset B that has no minimal element. Let C be the set of elements a of A such that $b \leqslant a$ for some b in B. Then C has no minimal element. Define g_1 and g_2 on A by

$$g_1(x) = 1; \qquad g_2(x) = \begin{cases} 2, & \text{if } x \in C; \\ 1, & \text{if } x \in A \smallsetminus C. \end{cases}$$

Define f from $\mathcal{P}(\{1, 2\})$ into $\{1, 2\}$ by

$$f(X) = \begin{cases} 2, & \text{if } 2 \in X; \\ 1, & \text{otherwise.} \end{cases}$$

Then both g_1 and g_2 are functions g such that $g(x) = f(g[\operatorname{pred}(x)])$. □

A well-founded total order is usually said to be a **well-ordered set.** So $(\mathbb{N}, <)$ is well-ordered. We now complete the circle of implications begun in the last two theorems.

THEOREM 1.4.9. *Well-founded strict orders admit induction.* □

For orders then, induction and recursion are equivalent. For iterative structures, they are not.

1.5. More building blocks

Note that we have *not* proved that $\mathbb{N}$ exists. We might supply this deficiency by offering the following **recursive definition,** namely,

(1) $1 \in \mathbb{N}$;
(2) if $n \in \mathbb{N}$, then $n1 \in \mathbb{N}$.

By $n1$ is meant the result of writing 1 to the right of n. So the natural numbers are obtained as **strings** $1 \cdots 1$, whose **entries** are vertical strokes. It is an exercise to check that the set of these strings is a model of the Peano axioms.

Note that our recursive definition of $\mathbb{N}$ is the recursive definition of a *set;* it should be distinguished from the recursive definition of a *function.*

If $A \subseteq \operatorname{dom}(f)$ and $f[A] \subseteq A$, then A is **closed** under f. One of the axioms of set-theory is that there is a set Ω that contains $\varnothing$ and is closed under the operation $x \mapsto x'$, where x' is the **(set-theoretic) successor** of x and is given by

$$x' = x \cup \{x\}.$$

So $(\Omega, \varnothing, ')$ is an iterative structure. The set ω is defined as the intersection of all subsets of Ω that contain $\varnothing$ and are closed under $x \mapsto x'$. Then $(\omega, \varnothing, ')$ satisfies the Peano axioms (exercise). Again we could offer a recursive definition:

(1) $\varnothing \in \omega$;
(2) if $n \in \omega$, then $n \cup \{n\} \in \omega$.

Normally $\varnothing$ is denoted by 0; and $0'$, by 1. In general, when $n \in \omega$, then

$$n = \{0, 1, \dots, n-1\}.$$

It is notationally convenient to consider the natural numbers to be the elements of ω, in the following way.

The set of all functions from a set A to a set B can be denoted by[4]

$$B^A. \tag{$*$}$$

If $n \in \omega$, then the nth **Cartesian power** of A is

$$A^n.$$

Thus, the nth Cartesian power of A is the set of functions from $\{0, 1, \dots, n-1\}$ to A. An element of A^n can be written as any one of

$$(a_0, \dots, a_{n-1}), \qquad i \longmapsto a_i, \qquad \vec{a}\,;$$

it can be called an **(ordered) n-tuple** from A. Note well that

$$A^0 = \{\varnothing\} = \{0\} = 1;$$

this is true even if A is empty. Also, every element of A^1 is $\{(0, a)\}$ for some a in A. So we have a bijection

$$x \longmapsto \{(0, x)\} \tag{$\dagger$}$$

from A to A^1. We may sometimes treat this bijection as an **identification:** that is, we may neglect to distinguish between a and $\{(0, a)\}$.

For any m and n in ω, we have a bijection

$$(\vec{x}\,, \vec{y}\,) \longmapsto \vec{x}\,^\frown\vec{y} \tag{$\ddagger$}$$

from $A^m \times A^n$ to A^{m+n}. In this notation, $\vec{a}\,^\frown\vec{b}$ is the $(m+n)$-tuple

$$(a_0, \dots, a_{m-1}, b_0, \dots, b_{n-1});$$

this is the $(m+n)$-tuple $\vec{c}$ such that

$$c_k = \begin{cases} a_k, & \text{if } k < m; \\ b_{k-m}, & \text{if } m \leqslant k < m+n. \end{cases}$$

We shall always treat the bijection in ($\ddagger$) as an identification; in particular, we shall always write $(\vec{a}\,, \vec{b}\,)$ instead of $\vec{a}\,^\frown\vec{b}$.

An **n-ary operation on** A is a function from A^n to A. The set of these is

$$A^{A^n}.$$

In particular, a 0-ary or **nullary operation** on A is an element of A^1; by the bijection in ($\dagger$) then, we may identify a nullary operation on A with an element of A.

An **n-ary relation on** A is a subset of A^n; the set of these is

$$\mathcal{P}(A^n).$$

[4]Some people write ${}^AB$ instead.

In particular, a nullary relation is a subset of A^0, that is, of 1 (or $\{0\}$); so the nullary relation is 0 or 1.

An n-ary operation on A is then a (certain kind of) subset of $A^n \times A$, and this product can be identified with $A^n \times A^1$ and hence with A^{n+1}; so an n-ary operation on A can be thought of as an $(n+1)$-ary relation on A. More precisely, if $f\colon A^n \to A$, then one may refer to the $(n+1)$-ary relation

$$\{(\vec{x}\,, f(\vec{x}\,))\colon \vec{x}\, \in A^n\}$$

as the **graph** of f; but there is a bijection between graphs in this sense and operations.

1.6. Well-ordered sets and cardinalities

The notion of *cardinality* of sets can be developed with the help of well-ordered sets.

An injection h from one relational structure, (A, R), to another, (B, S), is an **embedding** if $x\ R\ y \iff h(x)\ S\ h(y)$. An **isomorphism** is then a surjective embedding. An **initial segment** of a relational structure is a subset that contains all of the predecessors of its elements. The initial segment is **proper** if it is not the whole order; otherwise it is **improper.** Only initial segments of *orders* will be of interest to us.

LEMMA 1.6.1. *Of any two well-ordered sets, one is isomorphic to an initial segment of the other.*

PROOF. Let $\mathfrak{A}$ and $\mathfrak{B}$ be well-ordered sets, and let ∞ be a non-element of B. Define h from A to $B \cup \{\infty\}$ by

$$h(x) = \begin{cases} \min(B \smallsetminus h[\operatorname{pred}(x)]), & \text{if this exists;} \\ \infty, & \text{otherwise.} \end{cases}$$

Let $A^* = \{x \in A\colon h(x) \neq \infty\}$ and $h^* = h \restriction A^*$. Then A^* is an initial segment of $\mathfrak{A}$, and h^* is an isomorphism between this and an initial segment of $\mathfrak{B}$. Moreover, one of these segments is improper. □

LEMMA 1.6.2. *If $\mathfrak{A}$ and $\mathfrak{B}$ are well-ordered sets, and $\mathfrak{A}$ is isomorphic to a proper initial segment of $\mathfrak{B}$, then $\mathfrak{B}$ is not isomorphic to an initial segment of $\mathfrak{A}$.* □

A set is **transitive** if it properly includes each of its elements. So A is transitive if and only if $x \in A \iff x \subset A$, that is,

$$x \in A\ \&\ y \in x \implies y \in A.$$

LEMMA 1.6.3. *On a well-ordered set $\mathfrak{A}$, let h be defined by*

$$h(x) = h[\operatorname{pred}(x)].$$

Then the image of h is transitive, is well-ordered by membership ($\in$), and, with this ordering, is isomorphic to $\mathfrak{A}$. □

A set is an **ordinal** if it is transitive and well-ordered by membership. Ordinals are often denoted by small letters from the beginning of the Greek alphabet, as α and β.

THEOREM 1.6.1. *Every member of an ordinal is an ordinal.* On *an ordinal, membership is the same as proper inclusion ($\subset$). The set-theoretic successor of an ordinal is an ordinal. The union of a set of ordinals is an ordinal.* □

THEOREM 1.6.2 (Burali-Forti Paradox). *The class of ordinals is transitive and well-ordered by membership, which is the same as proper inclusion.* □

The paradox is that, if the class of ordinals is a **set,** then it is an ordinal, so it belongs to itself and therefore properly includes itself, which is absurd. So the class of ordinals is *not* a set; it is 'too big' to be a set; it is a **proper class.** The class of ordinals can be understood to have the following recursive definition.

(1) $\varnothing$ is an ordinal;
(2) if x is an ordinal, then so is x';
(3) the union of a set of ordinals is an ordinal.

In particular, ω and its elements are ordinals.

If $\mathfrak{A}$ is a well-ordered set, then, by the lemmas, it is isomorphic to a unique ordinal. Two sets have the same **cardinality** if there is a bijection between them. One of the equivalent forms of the **Axiom of Choice** is that every set can be well-ordered. Therefore every set has the same cardinality as some ordinal. The least ordinal with the same cardinality as the set is the **cardinality** of the set. The cardinality of A is denoted by

$$|A|.$$

An ordinal that is the cardinality of some set is a **cardinal.** We have $|A| \leqslant |\mathcal{P}(A)|$, but there is no bijection between A and $\mathcal{P}(A)$; therefore $|A| < |\mathcal{P}(A)|$. In particular, there is no largest cardinality. This allows us to define, on the class of ordinals, the function

$$\alpha \longmapsto \aleph_\alpha,$$

where $\aleph_\alpha$ is the least infinite cardinal greater than those $\aleph_\beta$ such that $\beta < \alpha$. In particular,

$$\aleph_0 = \omega.$$

All of the finite ordinals are cardinals. These and $\aleph_0$ are **countable;** the other cardinals are **uncountable.**

1.7. Structures

An informal definition of structure was given in 1.2; now we can give more formal definition. A **structure** is an ordered pair $(A, \mathcal{I})$, also referred to as $\mathfrak{A}$, where:

(1) A is a set, called the **universe** of the structure;
(2) $\mathcal{I}$ is a function, written also

$$s \longmapsto s^{\mathfrak{A}},$$

whose domain $\mathcal{L}$ is called the **signature** of the structure;
(3) $s^{\mathfrak{A}}$ is either an element of A or an n-ary operation or relation on A for some positive n, for each s in $\mathcal{L}$.

Then $\mathfrak{A}$ is more precisely an **$\mathcal{L}$-structure.** or a structure **of** $\mathcal{L}$. If $\mathcal{L} = \{s_0, s_1, \dots\}$, then $\mathfrak{A}$ can be written

$$(A, {s_0}^{\mathfrak{A}}, {s_1}^{\mathfrak{A}}, \dots).$$

The elements, operations, and relations $s^{\mathfrak{A}}$ may be called **basic.**

We have made use of the inductive structure $(\omega, 0, ')$ to define Cartesian powers and hence structures in general. Algebra provides a wealth of examples of structures:

(1) a group G, or $(G, \cdot, {}^{-1}, 1)$;
(2) an *abelian* group G, or $(G, +, -, 0)$;

(3) a unital ring R, or $(R, +, -, \cdot, 0, 1)$;
(4) the ring $\mathbb{Z}$, or $(\mathbb{Z}, +, -, \cdot, 0, 1)$;
(5) the field $\mathbb{R}$, or $(\mathbb{R}, +, -, \cdot, 0, 1)$;
(6) the two-element field $\mathbb{F}_2$, or $(\mathbb{F}_2, +, -, \cdot, 0, 1)$;
(7) a vector-space V over a field K; here the signature of V is

$$\{+, -, 0\} \cup \{a\,\cdot : a \in K\},$$

where $a\,\cdot$ is the singular operation of multiplying by a.

Further examples include:

(7) an order $(\Omega, <)$;
(8) the *ordered* field $\mathbb{R}$, or $(\mathbb{R}, +, -, \cdot, 0, 1, <)$.

From a set Ω arises what we might call the **power-set** structure on Ω, namely

$$(\mathcal{P}(\Omega), \cap, \cup, {}^{\mathrm{c}}, \varnothing, \Omega, \subseteq). \tag{$*$}$$

In case Ω is the 1-element set $\{0\}$, which is 1, then we have $\mathcal{P}(\Omega) = \{0, 1\}$, and we may write out the structure in $(*)$ as

$$(\mathbb{B}, \&, \vee, \neg, 0, 1, \vDash).$$

In particular, $\mathbb{B} = \{0, 1\}$. I propose to refer to any structure with universe $\mathbb{B}$ as a **truth-structure.** In this context, we can understand 1 as truth, and 0 as falsehood. **Propositional logic** is the study of truth-structures.

With $\mathcal{I}$ as above in the arbitrary structure $(A, \mathcal{I})$:

(1) $s^{\mathfrak{A}}$ is the **interpretation** in $\mathfrak{A}$ of s;
(2) s is a **symbol** for $s^{\mathfrak{A}}$.

So s is one of the following:

(1) a **constant,** if $s^{\mathfrak{A}}$ is an element of A;
(2) an n**-ary function-symbol,** if $s^{\mathfrak{A}}$ is an n-ary operation on A;
(3) an n**-ary predicate,**[5] if $s^{\mathfrak{A}}$ is an n-ary relation on A.

Since nullary operations on A can be considered as elements of A, a constant can be considered as a nullary function-symbol.

Suppose $\mathfrak{A}$ and $\mathfrak{B}$ are two structures with the same signature $\mathcal{L}$. A **homomorphism** from $\mathfrak{A}$ to $\mathfrak{B}$ is a function h from A to B such that

(1) $h(c^{\mathfrak{A}}) = c^{\mathfrak{B}}$ for all constants c in $\mathcal{L}$;
(2) $h(f^{\mathfrak{A}}(a_0, \dots, a_{n-1})) = f^{\mathfrak{B}}(h(a_0), \dots, h(a_{n-1}))$ for all a_i in A and n-ary function-symbols f in $\mathcal{L}$, for all positive n;
(3) $(a_0, \dots, a_{n-1}) \in R^{\mathfrak{A}} \implies (h(a_0), \dots, h(a_{n-1})) \in R^{\mathfrak{B}}$ for all a_i in A, for all n-ary predicates R in $\mathcal{L}$, for all positive n.

To say that h is a homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$, we may write

$$h \colon \mathfrak{A} \longrightarrow \mathfrak{B}.$$

Then h is an **isomorphism** if it is a bijection and its inverse is a homorphism. If $A \subseteq B$, and the inclusion of A in B is a homomorphism, then we write

$$\mathfrak{A} \subseteq \mathfrak{B}$$

[5]Or *relation-symbol.*

and say that $\mathfrak{A}$ is a **substructure** of $\mathfrak{B}$. So a substructure of $\mathfrak{B}$ is a structure whose universe is a subset of B that is **closed** under the basic operations of $\mathfrak{B}$ (including the nullary operations).

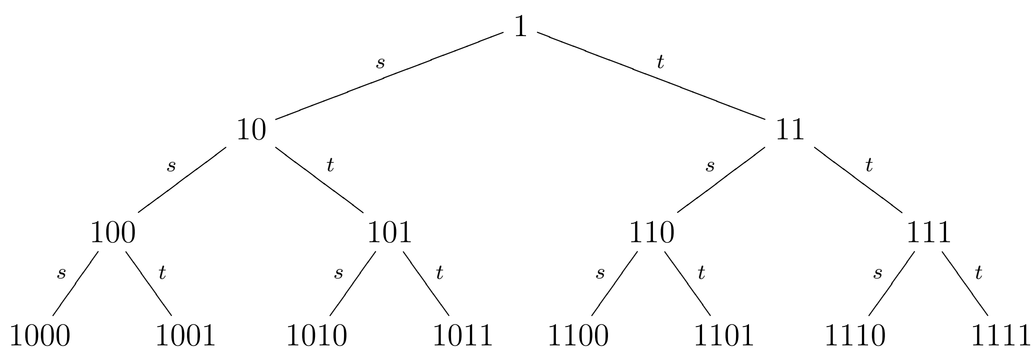
1.8. Algebras

A structure in a signature with no predicates is an **algebra,** and the signature itself may be called **algebraic.** In particular, iterative structures in the sense of §1.4 are algebras. A substructure of an algebra can be called a **subalgebra.** An arbitrary algebra admits **(proof by) induction** if it has no proper subalgebras. An algebra $\mathfrak{A}$ in signature $\mathcal{L}$ admits **recursion** if, for each algebra $\mathfrak{B}$ of $\mathcal{L}$, there is a unique homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$. The earlier definitions, for iterative structures, were just special cases of these.

There are also two 'degenerate' cases to consider. If there are no constants, then only the empty structure admits induction or recursion. If there are no function-symbols (of positive arity), an algebra admits induction if and only if its universe consists entirely of interpretations of the constants; the algebra admits recursion if and only if, in addition, distinct constants have distinct interpretations.

So the simplest *interesting* cases of algebras that admit induction or recursion are just the ones we have already considered, namely the iterative structures, in a signature consisting of one constant and one singulary function-symbol.

In the signature $\{1, s, t\}$ with one constant and *two* singulary function-symbols, let $\mathfrak{A}$ be the structure whose universe comprises the binary numerals (starting with 1), where $1^{\mathfrak{A}}$ is 1, and $s^{\mathfrak{A}}$ is adding 0 on the right, and $t^{\mathfrak{A}}$ is adding 1 on the right. Then $1110 = s^{\mathfrak{A}}(t^{\mathfrak{A}}(t^{\mathfrak{A}}(1))) = s(t(t(1))) = 1^{tts}$, and (part of) $\mathfrak{A}$ might be depicted as follows.



This structure admits induction, since every numeral can be obtained from 1 by application of $s^{\mathfrak{A}}$ and $t^{\mathfrak{A}}$; it admits recursion, since every numeral is so obtained in a unique way.

Theorem 1.8.1 (Recursion). *An algebra admits recursion, provided:*

(1) it admits induction,
(2) its basic operations are injections, and
(3) the ranges of these operations (including the nullary operations) are pairwise disjoint.

Proof. Suppose $\mathfrak{A}$ is an algebra, in a signature $\mathcal{L}$, meeting the given conditions. Let $\mathfrak{B}$ be another $\mathcal{L}$-structure. Let $\mathbf{C}$ be the set of all relations D from A to B such that, if $a\ D\ b$, then either $a = c^{\mathfrak{A}}$ and $b = c^{\mathfrak{B}}$ for some constant c in $\mathcal{L}$, or $a = f^{\mathfrak{A}}(d_0, \dots, d_{n-1})$ and $b = f^{\mathfrak{B}}(e_0, \dots, e_{n-1})$ for some n-ary function-symbol in $\mathcal{L}$, for some positive n, where

$d_i\ D\ e_i$ for each i in n. Then $\bigcup \mathbf{C}$ is a homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$ (exercise); it is unique by induction (exercise). □

The converse to this theorem, in the signature $\{1, \mathrm{s}\}$, is Theorem 1.4.3. The proof has three parts: (1) all algebras in the signature that admit recursion are isomorphic; (2) there is a particular algebra, namely $(\mathbb{N}, 1, \mathrm{s})$, that admits recursion *and* satisfies the hypotheses of Theorem 1.8.1; (3) therefore all algebras in the signature that admit recursion must satisfy the hypotheses. To follow this line of argument in arbitrary algebraic signatures, we need to find, in every such signature, an example of an algebra that admits recursion. A first step in this direction is the following.

THEOREM 1.8.2 (Induction). *Every algebra has a unique subalgebra that admits induction.*

PROOF. Let $\mathfrak{A}$ be an algebra in a signature $\mathcal{L}$. The set of subalgebras of $\mathfrak{A}$ is nonempty and ordered by the substructure-relation $\subseteq$. If $\mathbf{C}$ is a set of subalgebras of $\mathfrak{A}$, then $\bigcap \mathbf{C}$ is also a subalgebra of $\mathfrak{A}$ (exercise). Therefore the intersection of the set of all subalgebras of $\mathfrak{A}$ is a subalgebra $\mathfrak{B}$ of $\mathfrak{A}$. Then $\mathfrak{B}$ has no proper subalgebras, so it admits induction; and it is a subalgebra of every subalgebra of $\mathfrak{A}$, so it is the only subalgebra of $\mathfrak{A}$ that admits induction. □

The subalgebra $\mathfrak{B}$ found in the proof can be understood as given by the following **recursive definition:**

(1) $c^{\mathfrak{A}} \in B$ whenever c is a constant in $\mathcal{L}$;
(2) for all positive n, for all n-ary function-symbols f in $\mathcal{L}$, if $\vec{a} \in B^n$, then $f^{\mathfrak{A}}(\vec{a}\,) \in B$.

For every algebraic signature $\mathcal{L}$, there is an algebra $\mathfrak{A}$ in $\mathcal{L}$ whose universe A is the set of all strings of symbols from $\mathcal{L}$, and where

(1) $c^{\mathfrak{A}}$ is just c, when c is a constant in $\mathcal{L}$;
(2) when f is an n-ary function-symbol of $\mathcal{L}$, then $f^{\mathfrak{A}}$ is the function that, from an n-tuple $(\mathbf{A}_0, \dots, \mathbf{A}_{n-1})$ of strings in A, constructs the string $f\mathbf{A}_0 \cdots \mathbf{A}_{n-1}$.

Let the least subalgebra of $\mathfrak{A}$ be denoted by

$$\mathrm{Tm}^0(\mathcal{L});$$

this is the algebra of **constant terms** of $\mathcal{L}$.

An **initial segment** of a string is a string obtained by deleting some (or no) entries on the right. The initial segment is **proper** if it results from deleting at least one entry.

LEMMA 1.8.1. *No proper initial segment of an element of* $\mathrm{Tm}^0(\mathcal{L})$ *is an element of* $\mathrm{Tm}^0(\mathcal{L})$.

PROOF. We prove by induction that every element of $\mathrm{Tm}^0(\mathcal{L})$ neither *is* a proper initial segment of another element, nor *has* another element as a proper initial segment. This is true for all constants in $\mathcal{L}$, since all other terms start with function-symbols that are not constants. Suppose the claim is true for terms $t_0, \dots, t_{n-1}$, and f is an n-ary function-symbol in $\mathcal{L}$. Suppose the term $ft_0 \cdots t_{n-1}$ is a proper initial segment of some other term. This term must take the form $gu_0 \cdots u_{m-1}$. Then g is f, and there is some k such that either t_k is a proper initial segment of u_k, or the other way around. Either way contradicts the inductive hypothesis. There is a similar contradiction if some proper initial segment of $ft_0 \cdots t_{n-1}$ is a term. □

THEOREM 1.8.3. $\mathrm{Tm}^0(\mathcal{L})$ *admits recursion.*

PROOF. Use the Recursion Theorem (1.8.1). By construction, $\mathrm{Tm}^0(\mathcal{L})$ admits induction. Its basic operations are injective, since if $ft_0 \cdots t_{n-1}$ is the same term as $fu_0 \cdots u_{n-1}$, then each t_i must be the same as u_i, by the last lemma. The basic operations have disjoint images, since elements of the image of f all start with f. □

The converse of the Recursion Theorem (1.8.1) now follows in the manner suggested. Moreover, another method of proving the Theorem itself now arises: If $\mathfrak{A}$ and $\mathfrak{B}$ are algebras with the same signature, then the product algebra $\mathfrak{A} \times \mathfrak{B}$ can be defined in the obvious way. If $\mathfrak{C}$ is the subalgebra of this that admits induction, and if $\mathfrak{A}$ meets the conditions of the Recursion Theorem, then C is just a homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$.

If $\mathfrak{A}$ is an algebra with signature $\mathcal{L}$, then the interpretation $c \mapsto c^{\mathfrak{A}}$ in $\mathfrak{A}$ of the constants in $\mathcal{L}$ extends recursively to a function $t \mapsto t^{\mathfrak{A}}$ from $\mathrm{Tm}^0(\mathcal{L})$ into A, once we require

$$ft_0 \cdots t_{n-1}{}^{\mathfrak{A}} = f^{\mathfrak{A}}(t_0{}^{\mathfrak{A}}, \dots, t_{n-1}{}^{\mathfrak{A}}).$$

We now introduce a set $\{\mathsf{x}_k \colon k \in \omega\}$ of new symbols, to be called **(individual) variables.** We may add some or all of these to $\mathcal{L}$ as new constants. For $\mathrm{Tm}^0(\mathcal{L} \cup \{\mathsf{x}_0, \dots, \mathsf{x}_{n-1}\})$, we write

$$\mathrm{Tm}^n(\mathcal{L});$$

this is the set of n**-ary terms** of $\mathcal{L}$. The union of these sets is $\mathrm{Tm}(\mathcal{L})$. If t is an n-ary term, and $\vec{a}$ is an n-tuple from an $\mathcal{L}$-structure $\mathfrak{A}$, then we recursively obtain an element $t^{\mathfrak{A}}(\vec{a}\,)$ of A as follows:

(1) $\mathsf{x}_k{}^{\mathfrak{A}}(\vec{a}\,) = a_k$;
(2) $c^{\mathfrak{A}}(\vec{a}\,) = c^{\mathfrak{A}}$;
(3) $(ft_0 \cdots t_{n-1})^{\mathfrak{A}}(\vec{a}\,) = f^{\mathfrak{A}}(t_0{}^{\mathfrak{A}}(\vec{a}\,), \dots, t_{k-1}{}^{\mathfrak{A}}(\vec{a}\,))$.

We shall see a special case of the function $t \mapsto t^{\mathfrak{A}}(\vec{a}\,)$ in §2.4 and then develop it more generally in §3.1.

1.9. Propositional logic

The function-symbols in the signature of a truth-structure can be called **propositional connectives.**[6] Possibilities include

(1) the nullary connectives 0 and 1;
(2) the singular connective $\neg$;
(3) the binary connectives &, $\vee$, $\Rightarrow$, $\Leftrightarrow$, and $\not\Leftrightarrow$.

Each of these has a standard interpretation as an operation on $\mathbb{B}$. The interpretations of connectives with positive arity can be given by **truth-tables:**

P	$\neg P$
0	1
1	0

P	Q	$P \mathbin{\&} Q$	$P \vee Q$	$P \Rightarrow Q$	$P \Leftrightarrow Q$	$P \not\Leftrightarrow Q$
0	0	0	0	1	1	0
1	0	0	1	0	0	1
0	1	0	1	1	0	1
1	1	1	1	1	1	0

There is an alternative approach to truth-structures. We can first understand $\mathbb{B}$ as the two-element field $\mathbb{F}_2$, with the following addition- and multiplication-tables.

[6]Alternatively, they are *Boolean connectives.*

$+$	0	1
0	0	1
1	1	0

$\cdot$	0	1
0	0	0
1	0	1

Then $\nLeftrightarrow$ is another symbol for addition on this field; & is another symbol for multiplication; and the remaining connectives are as follows.

symbol	interpretation
$\neg$	$x \mapsto x + 1$
$\vee$	$(x, y) \mapsto x \cdot y + x + y$
$\Rightarrow$	$(x, y) \mapsto x \cdot y + x + 1$
$\Leftrightarrow$	$(x, y) \mapsto x + y + 1$

As mentioned above (p. 19), propositional logic is the study of truth-structures. A *particular* propositional logic, or a **propositional calculus,** consists of:

(1) for each n in ω, a set of strings called n-ary **(propositional) formulas;**
(2) a function $\mathbf{F} \mapsto \widehat{\mathbf{F}}$ that converts each n-ary propositional formula into an n-ary operation on $\mathbb{B}$;
(3) a set of formulas called **axioms;**
(4) some operations, called **rules of inference,** on the set of all formulas.

The axioms and rules of inference together constitute a **proof system.** Desirable features of a propositional logic include the following.

The set of formulas should be defined *recursively,* so that it admits proof by *induction.* Moreover, it should admit recursion itself. One way to achieve this, as shown in the previous section, is to let formulas be *terms* in the sense defined there; but there are alternatives.

The function $\mathbf{F} \mapsto \widehat{\mathbf{F}}$, the set of axioms, and the rules of inference should then be recursively defined.

For every n, for every n-ary operation g on $\mathbb{B}$, there should be some k and some $(n + k)$-ary formula $\mathbf{F}$ such that

$$\widehat{\mathbf{F}}(\vec{x}, \vec{y}) = g(\vec{x}).$$

In particular, if $i < n$, then the **projection** $\vec{x} \mapsto x_i$ from $\mathbb{B}^n$ to $\mathbb{B}$ will be $\widehat{\mathbf{F}}$ for some formula $\mathbf{F}$; most naturally, this formula is just P_i, a **propositional variable.** (This is a special case of the individual variables introduced in the last section.)

There will be propositional connectives, as mentioned above. These need not be formulas by themselves; but if $*$ is an n-ary connective, then there is a corresponding interpretation $\widehat{*}$ as an n-ary operation on $\mathbb{B}$. If $\mathbf{F}_0, \dots, \mathbf{F}_{n-1}$ are m-ary formulas, then there should be an m-ary formula $\mathbf{G}$ such that

$$\widehat{\mathbf{G}} = \widehat{*} \circ (\widehat{\mathbf{F}}_0, \dots, \widehat{\mathbf{F}}_{n-1}).$$

This $\mathbf{G}$ will presumably be some string in which $*$ appears as an entry, and in which the $\mathbf{F}_i$ appear as substrings.

If a substring of a formula is also a formula, and it is replaced by another formula, then the result should still be a formula, and that in a 'natural' way.

If $\widehat{\mathbf{F}}$ is constantly 1, then $\mathbf{F}$ is a **tautology.** It is desirable that all axioms be tautologies, and that the set of tautologies be closed under the rules of inference. Moreover, it is desirable that the set of tautologies be *least* with these properties. Then the set of tautologies will be recursively defined and so admit induction.

In general, *logic* begins as a way to understand ordinary language and to make it precise. Propositional connectives correspond to conjunctions and other 'structural' words like *and, or, not,* and *if... then.* For example, we interpret the connectives $\neg$ and $\Rightarrow$ as in the truth-tables above, because:

(1) we think[7] of 0 as falsity and 1 as truth;
(2) we take $\neg$ to stand for a word like *not* that *negates* sentences, and we take $\Rightarrow$ to stand for the locution *if... then;*
(3) in our mathematical writing at any rate,
  (a) a claim will be true if and only if its negation is false, and
  (b) an implication *If A, then B* will be false if and only if A is true, but B is false.

The function $\mathbf{F} \mapsto \widehat{\mathbf{F}}$ assigns a 'meaning' to formulas. Hence anything to do with this function can be called **semantic.** By contrast, a proof system is **syntactic,** involving formulas only as strings. (The etymologies of *semantic* and *syntactic* are discussed in Appendix D.) Gottlob Frege is credited with the first proof system. A bit of his peculiar notation (discussed in Appendix E) survives: If $\mathbf{F}$ is an axiom or can be obtained from the axioms by (possibly repeated) application of the rules of inference, then we write

$$\vdash \mathbf{F},$$

apparently borrowing from Frege's notation. By contrast, if $\mathbf{F}$ is a tautology, then we may write

$$\vDash \mathbf{F}.$$

It is easy to ensure that $\vdash \mathbf{F}$ implies $\vDash \mathbf{F}$. Forty-two years after Frege, in 1921, Emil Post published a proof [**28**, p. 169] that there are proof systems in which $\vdash \mathbf{F}$ if and only if $\vDash \mathbf{F}$.

Exercises

Exercise 1.1. Suppose $\mathfrak{A}$ is an inductive structure, and $\mathfrak{B}$ is another structure in the signature $\{1, \mathrm{s}\}$, where $1^{\mathfrak{A}} = 1^{\mathfrak{B}}$, and the two functions $\mathrm{s}^{\mathfrak{A}}$ and $\mathrm{s}^{\mathfrak{B}}$ agree on the intersection $A \cap B$ of their domains (that is, $\mathrm{s}^{\mathfrak{A}} \upharpoonright A \cap B = \mathrm{s}^{\mathfrak{B}} \upharpoonright A \cap B$). Prove that $A \subseteq B$.

Exercise 1.2. Prove Theorem 1.4.2 by obtaining f as an *intersection* of relations from A to B.

Exercise 1.3.

(1) Find a relational structure $(A, <)$ that admits induction, although $<$ is not transitive.
(2) Prove Lemma 1.4.1.
(3) Find a *partial* order that admits induction.

Exercise 1.4. Prove Lemma 1.4.2.

Exercise 1.5. Prove Theorem 1.4.9.

Exercise 1.6. Prove that $\mathbb{N}$ as defined in §1.5 is indeed a model of the Peano axioms.

Exercise 1.7. Prove that ω is a model of the Peano axioms.

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[7]It is possible to think the other way, where 0 is truth and 1 is falsity; this is done, for example, in [**32**, Ch. 4, Exercise 3.7, p. 178].

Exercise 1.8. Supply all missing details in §1.6.

Exercise 1.9. Verify that the definition of isomorphism given in §1.6 for relational structures agrees with that given in §1.7 for arbitrary structures.

Exercise 1.10. Supply the missing details in the proof of Theorem 1.8.1.

Exercise 1.11. Prove that the intersection of a set of subalgebras of an algebra is a subalgebra.

Exercise 1.12. Fill in the details of the alternative proof of the Recursion Theorem (1.8.1) mentioned after Theorem 1.8.3.

CHAPTER 2

Propositional model theory

2.1. Propositional formulas

This chapter presents a kind of model theory of propositional logic. It is inspired in part by Chang and Kiesler [**6**, § 1.2], who describe the subject as '"toy" model theory'. In this toy model theory, the role of structures will by played by *truth-assignments.* These will provide interpretations for propositional formulas and will serve as models for sets of propositional formulas.

Until §2.9, the official signature for our propositional logic will be $\{\neg, \Rightarrow\}$. Our **propositional variables** will compose the set $\{P_0, P_1, \dots\}$, or $\{P_n \colon n \in \omega\}$; we may also denote this set by

$$V.$$

To denote arbitrary members of V, we may use the boldface letters $\mathbf{P}$, $\mathbf{Q}$, and $\mathbf{R}$. These are in boldface as a reminder that they are not *themselves* propositional variables.[1] The set of **propositional formulas** will be called

$$\mathrm{PF}.$$

We give this a recursive definition:

(1) Every propositional variable belongs to PF;
(2) if $\mathbf{A}$ belongs to PF, then so does $\neg\mathbf{A}$;
(3) if $\mathbf{A}$ and $\mathbf{B}$ belong to PF, then so does $(\mathbf{A} \Rightarrow \mathbf{B})$.

So the propositional formulas are among the strings, each of whose entries is

(1) an element of the set V of variables, or
(2) one of the connectives $\neg$ or $\Rightarrow$, or
(3) one of the parentheses (or).

We may refer to an arbitrary such string by $\mathbf{A}$ or $\mathbf{B}$, as we did in the definition of PF; we may refer to a *formula* by $\mathbf{F}$, $\mathbf{G}$, $\mathbf{H}$, or $\mathbf{K}$.

A formula obtained as $\neg\mathbf{F}$ is a **negation;** as $(\mathbf{F} \Rightarrow \mathbf{G})$, an **implication.** So negations begin with the negation-sign $\neg$; implications, with the left parenthesis (. Every other formula is simply a variable.

We may also refer to the *operations* of forming $\neg\mathbf{A}$ from $\mathbf{A}$, and $(\mathbf{A} \Rightarrow \mathbf{B})$ from $(\mathbf{A}, \mathbf{B})$, as negation and implication respectively. We may denote the operation of negation by neg; of implication, by imp. Then $(\mathrm{PF}, P_0, P_1, P_2, \dots, \mathrm{neg}, \mathrm{imp})$ is the algebra admitting induction whose existence is guaranteed by the Induction Theorem (1.8.2).

[1]In technical terms, they are *syntactical variables.* That is, they are certain symbols of the *syntax language.* This is the language—English, with some extra symbols—that we are using now. We are using this syntax language to talk about the *object language,*which in this case is the language of propositional logic. See [**8**, § 8].

By induction one can prove for example that every propositional formula has the same number of left as right parentheses. A somewhat more interesting induction will prove Theorem 2.1.1 below.

Suppose $g\colon V \to \mathrm{PF}$. We can use this as a basis for substituting formulas for variables in a formula. Indeed, suppose $\mathbf{F}$ is an **n-ary formula,** so that its variables appear in the list $(P_0, \dots, P_{n-1})$. Then we can denote $\mathbf{F}$ more precisely by

$$\mathbf{F}(P_0, \dots, P_{n-1}).$$

Suppose $g(P_k)$ is $\mathbf{G}_k$ for each k in ω. If we go through $\mathbf{F}$ entry by entry, replacing each variable P_k with the formula $\mathbf{G}_k$, then the resulting string can be denoted by

$$\mathbf{F}(\mathbf{G}_0, \dots, \mathbf{G}_{n-1}),$$

or simply by

$$\mathbf{F}(g).$$

This is the result of **substitution with respect to g.** If exactly m entries in $\mathbf{F}$ are variables, so that $\mathbf{F}$ can be written as

$$\dots P_{k_0} \dots P_{k_1} \dots \cdots \dots P_{k_{m-1}} \dots,$$

then the formula $\mathbf{F}(g)$ is

$$\dots \mathbf{G}_{k_0} \dots \mathbf{G}_{k_1} \dots \cdots \dots \mathbf{G}_{k_{m-1}} \dots$$

THEOREM 2.1.1. *If $g\colon V \to \mathrm{PF}$, and $\mathbf{F}$ is in PF, then so is $\mathbf{F}(g)$.*

PROOF. By induction on formulas, we prove that the set of formulas $\mathbf{F}$ such that $\mathbf{F}(g)$ is a formula is all of PF.

(1) If $\mathbf{P}$ is a variable, then $g(\mathbf{P})$ is assumed to be a formula; but $\mathbf{P}(g)$ is $g(\mathbf{P})$, so $\mathbf{P}(g)$ is in PF.
(2) Suppose $\mathbf{F}(g)$ is a formula $\mathbf{H}$. Then substitution with respect to g in $\neg\mathbf{F}$ results in $\neg\mathbf{H}$, which is in PF by its definition.
(3) Suppose $\mathbf{F}(g)$ and $\mathbf{G}(g)$ are formulas $\mathbf{H}$ and $\mathbf{H}'$ respectively. Then $(\mathbf{F} \Rightarrow \mathbf{G})(g)$ is $(\mathbf{H} \Rightarrow \mathbf{H}')$, which again is in PF by definition.

This completes the induction and the proof. □

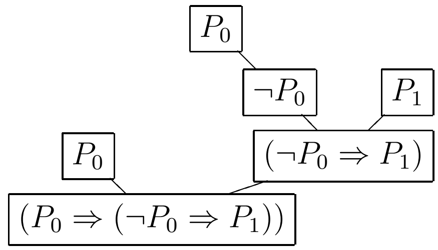
If the foregoing discussion of substitution seems too informal or imprecise, let it be noted that the operation $\mathbf{F} \mapsto \mathbf{F}(g)$ can be defined *recursively,* by means of Theorem 2.2.1 below. However, substitution makes sense for sets of strings that do *not* admit recursion or even induction.

2.2. Recursion

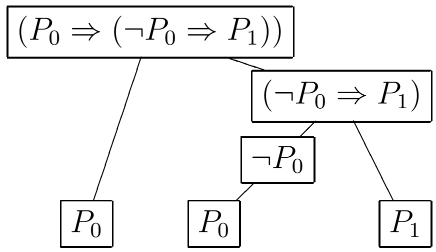
More is true than that PF is defined recursively, so that it admits proof by induction. Every propositional formula carries the history of its construction, which can be displayed in a *tree* whose 'trunk' or 'root' is the formula, and whose 'leaves' are variables. For example, the formula $(P_0 \Rightarrow (\neg P_0 \Rightarrow P_1))$ can be analyzed in the following way, up[2] to

[2]The English *analyze* is from the Greek ἀνάλυσις, which literally means a freeing *up.*

the variables.



We can also draw this picture upside down, showing the formula as built up from the variables.



By the formal definition, a **tree** is an order $(T, <)$ such that, for each element of T, the set of predecessors of that element is well-ordered by $<$. In the tree above, we have $\mathbf{F} < \mathbf{G}$ just in case $\mathbf{G}$ is a *subformula* of $\mathbf{F}$. Trees can be drawn for all formulas, because of Lemma 2.2.2 below. First, an analogue of Lemma 1.8.1 is

LEMMA 2.2.1. *No proper initial segment of a propositional formula is a formula.*

PROOF. We prove by induction that every formula neither *has* a proper initial segment that is a formula, nor *is* itself a proper initial segment of another formula.

By definition of PF, as we noted, every formula that is not just a variable starts with $\neg$ or (. So our claim holds for variables.

Suppose the claim holds for $\mathbf{F}$. Then it holds for $\neg\mathbf{F}$. Indeed, if $\neg\mathbf{F}$ has an initial segment that is a formula, then we can write this formula as $\neg\mathbf{H}$, where $\mathbf{H}$ is a formula. But $\mathbf{H}$ is an initial segment of $\mathbf{F}$, so by inductive hypothesis it must be $\mathbf{F}$ itself. Similarly, $\neg\mathbf{F}$ is not a proper initial segment of another formula.

Finally, suppose the claim holds for $\mathbf{F}$ and $\mathbf{G}$. Then it holds for $(\mathbf{F} \Rightarrow \mathbf{G})$. Indeed, suppose this has an initial segment that is a formula; then we can write this formula as $(\mathbf{H} \Rightarrow \mathbf{K})$ for some formulas $\mathbf{H}$ and $\mathbf{K}$. But then $\mathbf{H}$ is an initial segment of $\mathbf{F}$, or the other way around. Therefore $\mathbf{H}$ is $\mathbf{F}$. Hence $\mathbf{K}$ is an initial segment of $\mathbf{G}$, or the other way around, so $\mathbf{K}$ is $\mathbf{G}$. Similar considerations apply if $(\mathbf{F} \Rightarrow \mathbf{G})$ is an initial segment of a formula. □

LEMMA 2.2.2 (Unique Readability). *The operations* neg *and* imp *on* PF *are injective, and their images are disjoint from each other and from V.*

PROOF. We already know that the images are disjoint since negations start with $\neg$; implications, (. The operation of negation is injective, since if $\neg\mathbf{F}$ and $\neg\mathbf{G}$ are the same formula, then so are $\mathbf{F}$ and $\mathbf{G}$. Finally, if $(\mathbf{F} \Rightarrow \mathbf{G})$ and $(\mathbf{H} \Rightarrow \mathbf{K})$ are the same formula, then $\mathbf{F}$ is an initial segment of $\mathbf{H}$, or the other way around, so $\mathbf{F}$ and $\mathbf{H}$ are the same by Lemma 2.2.1, and hence so are $\mathbf{G}$ and $\mathbf{K}$; thus implication is injective. □

Hence an implication takes the form $(\mathbf{F} \Rightarrow \mathbf{G})$ for some *unique* formulas $\mathbf{F}$ and $\mathbf{G}$. We may refer to $\mathbf{F}$ as the **antecedent;** $\mathbf{G}$, the **consequent;** of the implication. By Theorem 1.8.1, we conclude

THEOREM 2.2.1 (Recursion). *Suppose A is a set, and*

(1) $h_0\colon V \to A$,
(2) g_1 is a singulary operation on A, and
(3) g_2 a binary operation on A.

Then there is a unique function h on PF *such that*

(1) h agrees with h_0 on V;
(2) $h(\neg\mathbf{F}) = g_1(h(\mathbf{F}))$, for all $\mathbf{F}$;
(3) $h((\mathbf{F} \Rightarrow \mathbf{G})) = g_2(h(\mathbf{F}), h(\mathbf{G}))$, for all $\mathbf{F}$ and $\mathbf{G}$. □

COROLLARY. *Suppose*

(1) $h_0\colon V \to A$,
(2) $g_1\colon \mathrm{PF} \times A \to A$, and
(3) $g_2\colon (\mathrm{PF} \times A)^2 \to A$.

Then there is a unique function h on PF *such that*

(1) h agrees with h_0 on V;
(2) $h(\neg\mathbf{F}) = g_1(\mathbf{F}, h(\mathbf{F}))$ for all $\mathbf{F}$;
(3) $h((\mathbf{F} \Rightarrow \mathbf{G})) = g_2((\mathbf{F}, h(\mathbf{F})), (\mathbf{G}, h(\mathbf{G})))$ for all $\mathbf{F}$ and $\mathbf{G}$. □

We used Unique Readability (Lemma 2.2.2) to obtain the Recursion Theorem (2.2.1) and its corollary. Conversely, Unique Readability follows from Recursion by the general method given in §1.8. It also follows directly from the corollary. Indeed, using the notation of this corollary, let A be PF, let h_0 and g_1 be chosen arbitrarily, and let g_2 be

$$((\mathbf{F}, \mathbf{F}'), (\mathbf{G}, \mathbf{G}')) \longmapsto \mathbf{F}.$$

Let h be the function guaranteed by the corollary. Then $h((\mathbf{F} \Rightarrow \mathbf{G})) = \mathbf{F}$. Thus h selects, from an implication, its antecedent. Since h is a function, the antecedent must be unique. Similarly for the consequent.

Note well that the Recursion Theorem is *not* a consequence of the Induction Theorem alone. For example, suppose we define PF without using parentheses. We shall still be able to use induction, but if we are not careful, we shall not have definitions by recursion. Indeed, say we define nPF (for 'not PF') so that:

(1) each variable is in nPF;
(2) if $\mathbf{A}$ is in nPF, then so is $\neg\mathbf{A}$;
(3) if $\mathbf{A}$ and $\mathbf{B}$ are in nPF, then so is $\mathbf{A} \Rightarrow \mathbf{B}$.

Then proof by induction in nPF is possible. However, suppose we try to define by recursion a function f from nPF into PF so as to send every element of the former to its 'equivalent' in the latter:

(1) $f(\mathbf{P}) = \mathbf{P}$;
(2) $f(\neg\mathbf{F}) = \neg f(\mathbf{F})$;
(3) $f(\mathbf{F} \Rightarrow \mathbf{G}) = (f(\mathbf{F}) \Rightarrow f(\mathbf{G}))$.

This *fails* as the definition of a function, since it implies that $f(P_0 \Rightarrow P_1 \Rightarrow P_2)$ must be both $(P_0 \Rightarrow (P_1 \Rightarrow P_2))$ and $((P_0 \Rightarrow P_1) \Rightarrow P_2)$, even though these are different formulas.

2.3. Notation

A correct way to avoid using parentheses is to use *Łukasiewicz* or *Polish notation*, writing $\Rightarrow \mathbf{F}\ \mathbf{G}$ instead of $(\mathbf{F} \Rightarrow \mathbf{G})$. This is just the notation used for terms in §1.8.

Alternatively, without changing the order of symbols, we can remove *some* parentheses from the formulas in PF, obtaining a set PF$'$ of formulas that still admits recursion. To be precise, every formula in PF$'$ will be a variable, a negation, or an implication. Then the recursive definition of PF$'$ is as follows.

(1) V is the set of variables in PF$'$.
(2) If $\mathbf{A}$ is a variable or a negation in PF$'$, then $\neg\mathbf{A}$ is a negation in PF$'$.
(3) If $\mathbf{A}$ is an implication in PF$'$, then $\neg(\mathbf{A})$ is a negation in PF.
(4) If $\mathbf{A}$ is a variable or a negation in PF$'$, and $\mathbf{B}$ is in PF$'$, then $\mathbf{A} \Rightarrow \mathbf{B}$ is an implication in PF$'$.
(5) If $\mathbf{A}$ is an implication in PF$'$, and $\mathbf{B}$ is in PF$'$, then $(\mathbf{A}) \Rightarrow \mathbf{B}$ is an implication in PF$'$.

Thus, no formula in PF$'$ is enclosed in parentheses; but an implication must be so enclosed when it is negated or used as the *antecedent* of another implication. It is left to the reader to formulate and prove an analogue of the Recursion Theorem (2.2.1), so that the following can then be proved:

THEOREM 2.3.1. *There is a unique bijection* $\mathbf{F} \mapsto \overline{\mathbf{F}}$ *from* PF *to* PF$'$ *such that*

(1) $\overline{\mathbf{P}} = \mathbf{P}$ *for all variables* $\mathbf{P}$*;*

(2) $\overline{\neg\mathbf{F}} = \begin{cases} \neg\overline{\mathbf{F}}, & \text{if } \mathbf{F} \text{ is a variable or negation;} \\ \neg(\overline{\mathbf{F}}), & \text{if } \mathbf{F} \text{ is an implication;} \end{cases}$

(3) $\overline{(\mathbf{F} \Rightarrow \mathbf{G})} = \begin{cases} \overline{\mathbf{F}} \Rightarrow \overline{\mathbf{G}}, & \text{if } \mathbf{F} \text{ is a variable or negation;} \\ (\overline{\mathbf{F}}) \Rightarrow \overline{\mathbf{G}}, & \text{if } \mathbf{F} \text{ is an implication.} \end{cases}$

The inverse of this function is a function $\mathbf{F} \mapsto \underline{\mathbf{F}}$ *from* PF$'$ *to* PF *such that*

(1) $\underline{\mathbf{P}} = \mathbf{P}$ *for all variables* $\mathbf{P}$*;*
(2) $\underline{\neg\mathbf{F}} = \neg\underline{\mathbf{F}}$*;*
(3) $\underline{\neg(\mathbf{F})} = \neg\underline{\mathbf{F}}$*;*
(4) $\underline{\mathbf{F} \Rightarrow \mathbf{G}} = (\underline{\mathbf{F}} \Rightarrow \underline{\mathbf{G}})$*;*
(5) $\underline{(\mathbf{F}) \Rightarrow \mathbf{G}} = (\underline{\mathbf{F}} \Rightarrow \underline{\mathbf{G}})$*.*

PROOF. In the notation of the corollary to the Recursion Theorem for Formulas, let A be the set of strings of the symbols in $V \cup \{\Rightarrow, \neg, (,)\}$, let h_0 be the inclusion of V in A, and let

$$g_1(\mathbf{F}, \mathbf{A}) = \begin{cases} \neg\mathbf{A}, & \text{if } \mathbf{F} \text{ is a variable or negation;} \\ \neg(\mathbf{A}), & \text{if } \mathbf{F} \text{ is an implication;} \end{cases}$$

$$g_2((\mathbf{F}, \mathbf{A}), (\mathbf{G}, \mathbf{B})) = \begin{cases} \mathbf{A} \Rightarrow \mathbf{B}, & \text{if } \mathbf{F} \text{ is a variable or negation;} \\ (\mathbf{A}) \Rightarrow \mathbf{B}, & \text{if } \mathbf{F} \text{ is an implication.} \end{cases}$$

Then the function from PF to PF$'$ exists uniquely as desired. This function is bijective, with inverse as claimed (details are left to the reader). □

Henceforth we may use formulas PF$'$ to denote the corresponding formulas in PF. But the official formulas still belong to PF. This is an important point. Substitution in formulas in PF$'$ may have undesirable results. For example, in $P_0 \Rightarrow P_1$, if we substitute $P_0 \Rightarrow P_2$ for P_0, we get $P_0 \Rightarrow P_2 \Rightarrow P_1$, which corresponds to the formula $(P_0 \Rightarrow (P_2 \Rightarrow P_1))$ in PF; but this is not what we get by substituting $(P_0 \Rightarrow P_2)$ for P_0 in $(P_0 \Rightarrow P_1)$.

2.4. Truth

A **truth-assignment** is a function from V to $\mathbb{B}$. Let ε be such a function. It determines a substitution $\mathbf{F} \mapsto \mathbf{F}(\varepsilon)$ as in § 2.1, although 0 and 1 are not formulas in PF. By recursion, ε uniquely determines a function h on PF as follows.

$$h(\mathbf{P}) = \varepsilon(\mathbf{P});$$

$$h(\neg\mathbf{F}) = \begin{cases} 1, & \text{if } h(\mathbf{F}) = 0; \\ 0, & \text{if } h(\mathbf{F}) = 1; \end{cases}$$

$$h(\mathbf{F} \Rightarrow \mathbf{G}) = \begin{cases} 0, & \text{if } h(\mathbf{F}) = 1 \text{ and } h(\mathbf{G}) = 0; \\ 1, & \text{otherwise.} \end{cases}$$

Alternatively, using the considerations in §1.9, we have

$$h(\neg\mathbf{F}) = h(\mathbf{F}) + 1;$$
$$h(\mathbf{F} \Rightarrow \mathbf{G}) = h(\mathbf{F}) \cdot h(\mathbf{G}) + h(\mathbf{G}) + 1.$$

That is, h is the unique homomorphism from $(\mathrm{PF}, P_0, P_1, \dots, \mathrm{neg}, \mathrm{imp})$ into

$$(\mathbb{B}, \varepsilon(P_0), \varepsilon(P_1), \dots, x \mapsto x+1, (x,y) \mapsto x \cdot y + y + 1).$$

As such, it can be compared with the function $t \mapsto t^{\mathfrak{A}}(\vec{a}\,)$ defined in §1.8. We can denote the homomorphism h by

$$\mathbf{F} \longmapsto \widehat{\mathbf{F}}(\varepsilon).$$

If $\mathbf{F}$ is an n-ary formula, then $\widehat{\mathbf{F}}(\varepsilon)$ depends only on the n-tuple $(\varepsilon(P_0), \dots, \varepsilon(P_{n-1}))$. (This is obvious, but can be confirmed by induction on formulas.) Denoting this n-tuple more briefly by $\vec{e}$, we may write

$$\widehat{\mathbf{F}}(\vec{e}\,)$$

instead of $\widehat{\mathbf{F}}(\varepsilon)$, and we may refer to $\vec{e}$ as an n**-ary truth-assignment.** Then the n-ary operation

$$\vec{e} \longmapsto \widehat{\mathbf{F}}(\vec{e}\,),$$

or just $\widehat{\mathbf{F}}$, on $\mathbb{B}$ is the **interpretation** of $\mathbf{F}$. The number $\widehat{\mathbf{F}}(\vec{e}\,)$ is the **truth-value** of $\mathbf{F}$ with respect to ε or $\vec{e}$. In particular, $\mathbf{F}$ is **true in** ε (or $\vec{e}$) if $\widehat{\mathbf{F}}(\varepsilon) = 1$; otherwise, $\mathbf{F}$ is **false in** ε (or $\vec{e}$).

The truth-values of an n-ary formula $\mathbf{F}$ with respect to all n-ary truth-assignments can be given in a **truth-table** with 2^n rows and with one column for each entry in $\mathbf{F}$ that is not a parenthesis. For example, the table for $P_0 \Rightarrow \neg P_0 \Rightarrow P_1$ is the following.

P_0	$\Rightarrow$	$\neg$	P_0	$\Rightarrow$	P_1
0	1	1	0	0	0
1	1	0	1	1	0
1	1	0	1	1	0
1	1	0	1	1	1

In general, we can think of the rows as indexed by the numbers less than 2^n, written in binary notation. Indeed, let us define the elements $e_i^{(k)}$ of $\mathbb{B}$, where $i < n$ and $k < 2^n$, by

$$k = \sum_{i=0}^{n-1} e_i^{(k)} \cdot 2^i.$$

Then row k of the truth-table corresponds to the truth-assignment $(e_0^{(k)},\dots,e_{n-1}^{(k)})$. The corresponding truth-value for $\neg\mathbf{F}$ will be found in the column indexed by $\neg$; for $\mathbf{F}\Rightarrow\mathbf{G}$, by $\Rightarrow$. The truth-table for P_i is

$$\begin{matrix} e_i^{(0)} \\ \vdots \\ e_i^{(2^n-1)} \end{matrix}$$

The truth-table for $\neg\mathbf{F}$ is $S\ T$, where T is the table for $\mathbf{F}$, and, assuming the column of T giving the values of $\mathbf{F}$ is

$$\begin{matrix} f_0 \\ \vdots \\ f_{2^n-1} \end{matrix}$$

then S is the column

$$\begin{matrix} f_0+1 \\ \vdots \\ f_{2^n-1}+1 \end{matrix}$$

Similarly, the truth-table for $\mathbf{F}\Rightarrow\mathbf{G}$ takes the form $T_0\ S\ T_1$, where T_0 is the table for $\mathbf{F}$; T_1, for $\mathbf{G}$; and S gives the values of $\mathbf{F}\Rightarrow\mathbf{G}$.

The following may seem obvious, once it is understood.

THEOREM 2.4.1 (Associativity). *Suppose* $\mathbf{F}$ *is an* n*-ary formula, and* $\mathbf{H}$ *is a formula* $\mathbf{F}(\mathbf{G}_0,\dots,\mathbf{G}_{n-1})$*, and* $\vec{e}$ *and* $\vec{f}$ *are truth-assignments (of appropriate arity) such that*

$$\widehat{\mathbf{G}}_k(\vec{e}) = f_k$$

for each k *in* n*. Then*

$$\widehat{\mathbf{F}}(\vec{f}) = \widehat{\mathbf{H}}(\vec{e}).$$

PROOF. We use induction on $\mathbf{F}$. If $\mathbf{F}$ is a variable, then it is P_k for some k in n, so $\mathbf{H}$ is $\mathbf{G}_k$, and

$$\widehat{\mathbf{H}}(\vec{e}) = \widehat{\mathbf{G}}_k(\vec{e}) = f_k = \widehat{P}_k(\vec{f}) = \widehat{\mathbf{F}}(\vec{f}).$$

Suppose the claim is true when $\mathbf{F}$ is $\mathbf{F}_0$ or $\mathbf{F}_1$. If $\mathbf{F}$ is $\neg\mathbf{F}_0$, then $\mathbf{H}$ is $\neg\mathbf{H}_0$, where $\mathbf{H}_0$ is $\mathbf{F}_0(\mathbf{G}_0,\dots,\mathbf{G}_{n-1})$, so that

$$\begin{aligned} \widehat{\mathbf{H}}(\vec{e}) &= 1+\widehat{\mathbf{H}}_0(\vec{e}) \\ &= 1+\widehat{\mathbf{F}}_0(\vec{f}) && \text{[by inductive hypothesis]} \\ &= \widehat{\mathbf{F}}(\vec{f}). \end{aligned}$$

The remaining case, where $\mathbf{F}$ is $(\mathbf{F}_0\Rightarrow\mathbf{F}_1)$, is left to the reader. □

A formula is a **tautology** if it is true in every truth-assignment. The Associativity Theorem immediately yields

THEOREM 2.4.2 (Substitution). *If* $\mathbf{F}$ *is an* n*-ary tautology, then* $\mathbf{F}(\mathbf{G}_0,\dots,\mathbf{G}_{n-1})$ *is a tautology.* □

Two n-ary formulas $\mathbf{F}$ and $\mathbf{G}$ are **(logically) equivalent** if the operations $\widehat{\mathbf{F}}$ and $\widehat{\mathbf{G}}$ are the same. Suppose $\mathbf{F}$ appears as a **substring** of $\mathbf{H}$, so that $\mathbf{H}$ can be written as

$$\dots\mathbf{F}\dots$$

We might expect to be able to **replace F** with **G**, obtaining a new formula

$$\dots \mathbf{G} \dots$$

or $\mathbf{H}'$ such that, if **F** and **G** are equivalent, then so are **H** and $\mathbf{H}'$. However, this fails for PF$'$. For example, P_0, $P_0 \Rightarrow P_0$, and $\neg(P_0 \Rightarrow P_0)$ are formulas in PF$'$, but in the last, if we replace $P_0 \Rightarrow P_0$ with P_0, we get the non-formula $\neg(P_0)$.

In PF, replacement does work in the obvious way. First we define **subformulas** recursively: Every formula is a subformula of itself, and moreover,

(1) subformulas of **F** are subformulas of $\neg\mathbf{F}$;
(2) subformulas of **F** or **G** are subformulas of $(\mathbf{F} \Rightarrow \mathbf{G})$.

LEMMA 2.4.1. *In* PF*, a formula that is a substring of* **F** *is a subformula of* **F**. □

LEMMA 2.4.2. *In a formula of* PF*, if a subformula is replaced with another formula, the result is a formula.*

PROOF. The claim is trivially true when the original formula is a variable. Suppose it is true when the original formula is **F** or **G**. Then it is true for $\neg\mathbf{F}$ and $(\mathbf{F} \Rightarrow \mathbf{G})$ as well, since the proper subformulas of these are subformulas of **F** or **G**. □

THEOREM 2.4.3 (Replacement). *In* PF*, suppose* **F** *and* $\mathbf{F}'$ *are equivalent, and* **F** *is a subformula of* **G***, and* $\mathbf{G}'$ *is the result of replacing* **F** *in* **G** *with* $\mathbf{F}'$*. Then* **G** *and* $\mathbf{G}'$ *are equivalent.* □

2.5. Logical entailment

We can think of truth as a *relation* from $\mathbb{B}^V$ to PF, namely the **truth-relation,** $\vDash$, given by

$$\varepsilon \vDash \mathbf{F} \iff \widehat{\mathbf{F}}(\varepsilon) = 1. \tag{$*$}$$

The complement of $\vDash$ can be denoted by $\nvDash$. Hence we can express a fundamental fact as follows:

LEMMA 2.5.1. *For all truth-assignments* ε *and formulas* **F***,*

$$\varepsilon \vDash \mathbf{F} \iff \varepsilon \nvDash \neg\mathbf{F}. \tag{$\dagger$}$$

PROOF. Suppose $e \in \mathbb{B}$. Then $e = 1 \iff e \neq 0 \iff e + 1 = 0$. Let **G** be $\neg\mathbf{F}$. Then

$$\begin{aligned} \varepsilon \vDash \mathbf{F} &\iff \widehat{\mathbf{F}}(\varepsilon) = 1 \\ &\iff 1 + \widehat{\mathbf{F}}(\varepsilon) = 0 \\ &\iff \widehat{\mathbf{G}}(\varepsilon) = 0 \\ &\iff \widehat{\mathbf{G}}(\varepsilon) \neq 1 \\ &\iff \varepsilon \nvDash \mathbf{G} \\ &\iff \varepsilon \nvDash \neg\mathbf{F}, \end{aligned}$$

as desired. □

Immediately,

$$\varepsilon \nvDash \mathbf{F} \iff \varepsilon \vDash \neg\mathbf{F}.$$

From the truth-relation, we obtain three new functions, as follows. A **model** of a set of formulas is a truth-assignment in which every element of the set is true. If Σ is a set of formulas, let

$$\operatorname{Mod}(\Sigma)$$

be the set of its models. This is the set

$$\bigcap_{\mathbf{F}\in\Sigma}\{\varepsilon\in\mathbb{B}^V\colon\varepsilon\vDash\mathbf{F}\}.$$

We now have a function $\Sigma\mapsto\operatorname{Mod}(\Sigma)$ from $\mathcal{P}(\mathrm{PF})$ to $\mathcal{P}(\mathbb{B}^V)$. The **theory** of a set of truth-assignments is the set of formulas that are true in all of the truth-assignments. If E is a set of truth-assignments, let

$$\operatorname{Th}(E)$$

be its theory. This is the set

$$\bigcap_{\varepsilon\in E}\{\mathbf{F}\in\mathrm{PF}\colon\varepsilon\vDash\mathbf{F}\}.$$

So we have a function $E\mapsto\operatorname{Th}(E)$ from $\mathcal{P}(\mathbb{B}^V)$ to $\mathcal{P}(\mathrm{PF})$. The **logical consequences** of a set of formulas are the formulas that are true in every model of the set. The logical consequences of Σ compose a set

$$\operatorname{Con}(\Sigma).$$

This is the set

$$\bigcap_{\varepsilon\in\operatorname{Mod}(\Sigma)}\{\mathbf{F}\in\mathrm{PF}\colon\varepsilon\vDash\mathbf{F}\},$$

which is

$$\operatorname{Th}(\operatorname{Mod}(\Sigma)).$$

So we have a singulary operation $\Sigma\mapsto\operatorname{Con}(\Sigma)$ on $\mathcal{P}(\mathrm{PF})$. If T is a set of formulas that is the theory of *some* set of truth-assignments, then T can be called a **theory,** simply. If $\mathbf{F}$ is a logical consequence of Σ, we may say also that Σ **logically entails** $\mathbf{F}$. So we have several ways[3] of saying the same thing:

(1) $\mathbf{F}$ is a logical consequence of Σ;
(2) Σ logically entails $\mathbf{F}$;
(3) $\mathbf{F}\in\operatorname{Con}(\Sigma)$.

The logical consequences of $\varnothing$ are called **tautologies;** these are the formulas that are true in *every* truth-assignment.

Note well that the definition of logical entailment is not recursive. There is, at the moment, no obvious way to prove by induction that a given set of formulas contains all logical consequences of Σ (or even all tautologies).

LEMMA 2.5.2. *The functions $\Sigma\mapsto\operatorname{Mod}(\Sigma)$ and $E\mapsto\operatorname{Th}(E)$ are inclusion-reversing, that is,*

(1) $\Sigma\subseteq\Gamma\implies\operatorname{Mod}(\Gamma)\subseteq\operatorname{Mod}(\Sigma)$, *and*
(2) $D\subseteq E\implies\operatorname{Th}(E)\subseteq\operatorname{Th}(D)$.

The operations $\Sigma\mapsto\operatorname{Con}(\Sigma)$ and $E\mapsto\operatorname{Mod}(\operatorname{Th}(E))$ are increasing, that is,

(3) $\Sigma\subseteq\operatorname{Con}(\Sigma)$;
(4) $E\subseteq\operatorname{Mod}(\operatorname{Th}(E))$. □

[3]Another way might be $\Sigma\vDash\mathbf{F}$, as suggested in §1.7; but this should not be confused with the notation introduced in $(*)$, which has a different meaning.

THEOREM 2.5.1. *A subset* Σ *of* PF *is a theory if and only if*

$$\Sigma = \mathrm{Con}(\Sigma).$$

PROOF. If $\Sigma = \mathrm{Con}(\Sigma)$, then Σ is the theory of $\mathrm{Mod}(\Sigma)$. For the converse, note from the lemma that $\mathrm{Th}(E) \subseteq \mathrm{Con}(\mathrm{Th}(E))$ by (3), but

$$\mathrm{Con}(\mathrm{Th}(E)) = \mathrm{Th}(\mathrm{Mod}(\mathrm{Th}(E))) \subseteq \mathrm{Th}(E)$$

by (4), so $\mathrm{Th}(E) = \mathrm{Con}(\mathrm{Th}(E))$. □

See Appendix F for a discussion of the functions $\Sigma \mapsto \mathrm{Mod}(\Sigma)$ and $E \mapsto \mathrm{Con}(E)$ in general terms.

2.6. Compactness

A set of formulas with a model can be called **satisfiable.**

LEMMA 2.6.1. Σ *logically entails* $\mathbf{F}$ *if and only if* $\Sigma \cup \{\neg\mathbf{F}\}$ *is not satisfiable.*

PROOF. Suppose Σ does *not* logically entail $\mathbf{F}$. Then Σ has a model ε in which $\mathbf{F}$ is false. Hence $\varepsilon \vDash \neg\mathbf{F}$ by Lemma 2.5.1, so ε is a model of $\Sigma \cup \{\neg\mathbf{F}\}$. Suppose conversely that $\Sigma \cup \{\neg\mathbf{F}\}$ has a model. Then $\mathbf{F}$ is false in this model, again by Lemma 2.5.1, so $\mathbf{F}$ is not a logical consequence of Σ. □

A set of formulas whose every *finite* subset has a model can be called **finitely satisfiable.**

LEMMA 2.6.2. *If* Σ *is finitely satisfiable, then so is* $\Sigma \cup \{\mathbf{F}\}$ *or* $\Sigma \cup \{\neg\mathbf{F}\}$.

PROOF. Suppose Σ is finitely satisfiable, but $\Sigma \cup \{\mathbf{F}\}$ is not. Then there is a finite subset Γ of Σ such that $\Gamma \cup \{\mathbf{F}\}$ has no model. Then $\Gamma \cup \{\neg\neg\mathbf{F}\}$ has no model, so $\Gamma \vDash \neg\mathbf{F}$ by the last lemma. Say Λ is another finite subset of Σ. Then $\Gamma \cup \Lambda$ is also a finite subset of Σ, so it has a model, and $\neg\mathbf{F}$ is true in each of its models. Thus $\Lambda \cup \{\neg\mathbf{F}\}$ has a model, by Lemma 2.5.2. Hence $\Sigma \cup \{\mathbf{F}\}$ is finitely satisfiable. □

THEOREM 2.6.1 (Compactness). *Every finitely satisfiable set of formulas is satisfiable.*

PROOF. Let Σ be finitely satisfiable. By recursion (in the sense of Theorem 1.4.7), we first define a function $n \mapsto \mathbf{F}_n$ from ω into PF. Suppose $\{\mathbf{F}_k \colon k < n\}$ has been defined. We then let $\mathbf{F}_n$ be P_n, if $\Sigma \cup \{\mathbf{F}_k \colon k < n\} \cup \{P_n\}$ is finitely satisfiable; otherwise, $\mathbf{F}_n$ is $\neg P_n$. This completes the recursive definition.

We now observe by induction that every set $\Sigma \cup \{\mathbf{F}_k \colon k < n\}$ is finitely satisfiable. Indeed, it is true by assumption when $n = 0$; and if it is true when $n = m$, then it is true when $n = m + 1$, by the last lemma and the definition of the $\mathbf{F}_k$.

Now let ε be the truth-assignment given by

$$\varepsilon(P_k) = \begin{cases} 1, & \text{if } \mathbf{F}_k = P_k; \\ 0, & \text{if } \mathbf{F}_k = \neg P_k. \end{cases} \tag{$*$}$$

This is a model of Σ. Indeed, suppose $\mathbf{G} \in \Sigma$. Then $\mathbf{G}$ is n-ary for some n. The finite set $\{\mathbf{G}\} \cup \{\mathbf{F}_k \colon k < n\}$ has a model ζ. In particular, ζ must agree with ε on $\{P_k \colon k < n\}$ (why?); so $\varepsilon \vDash \mathbf{G}$. □

There are sets Σ of formulas such that *every* finite subset of Σ has a model that is not a model of Σ itself. For example, let Σ_n comprise the formulas

$$P_0 \Rightarrow P_1 \Rightarrow \cdots \Rightarrow P_k$$

where $k < n$. (The precise recursive definition of the sets Σ_n is left as an exercise.) So Σ_0 is empty, $\Sigma_1 = \{P_0\}$, and we have a chain

$$\Sigma_0 \subseteq \Sigma_1 \subseteq \Sigma_2 \subseteq \cdots .$$

Let $\Sigma = \bigcup_{n\in\omega} \Sigma_n$. Then every finite subset of Σ is a subset of some Σ_n. Let ε_n be the truth-assignment such that

$$\varepsilon_n(P_k) = 1 \iff k < n.$$

Then ε_n is a model of Σ_n, but not of Σ_{n+1} (why?), hence not of Σ. But Σ must have a model, by Compactness. In fact, $P_k \mapsto 0$ is a model.

If a set A is a *finite* subset of a set B, we may denote this by

$$A \subseteq_{\mathrm{f}} B.$$

Now one consequence of the Compactness Theorem can be expressed as follows:

Corollary. $\operatorname{Con}(\Sigma) = \displaystyle\bigcup_{\Gamma\subseteq_{\mathrm{f}}\Sigma} \operatorname{Con}(\Gamma).$

Proof. By Theorem 2.5.1, it is enough to show that

$$\operatorname{Con}(\Sigma) \subseteq \bigcup_{\Gamma\subseteq_{\mathrm{f}}\Sigma} \operatorname{Con}(\Gamma).$$

Suppose $\mathbf{F}$ is *not* a member of the union. Then, for each finite subset Γ of Σ, the set $\operatorname{Con}(\Gamma)$ does not contain $\mathbf{F}$, and so the set $\Gamma \cup \{\neg\mathbf{F}\}$ is satisfiable, by Lemma 2.6.1. This means $\Sigma \cup \{\neg\mathbf{F}\}$ is finitely satisfiable; so it is satisfiable, by the Compactness Theorem. Therefore $\neg\mathbf{F} \notin \operatorname{Con}(\Sigma)$, again by Lemma 2.6.1. □

2.7. Syntactic entailment

Logical entailment is one way to derive formulas from a given set of formulas. Another way is by *formal proof* or *deduction*.

A **proof system** consists of *axioms* and *rules of inference*. An **axiom** is a particular formula, and a **rule of inference** is a clearly defined way of obtaining one formula from finitely many others. Then a **formal proof** from a set Σ of formulas is a list of formulas, each of which is an element of Σ, or is an axiom, or is obtainable by a rule of inference from formulas appearing earlier on the list. The last formula on the list is the **conclusion;** the formulas in Σ are **hypotheses** from which this conclusion is **deducible** in the system. We may say also that the conclusion is a **syntactic consequence** of the hypotheses.

Our axioms will take any of the following three forms:

(1) $\mathbf{F} \Rightarrow \mathbf{G} \Rightarrow \mathbf{F}$ (**Affirmation of the Consequent**[4]),
(2) $(\mathbf{F} \Rightarrow \mathbf{G} \Rightarrow \mathbf{H}) \Rightarrow (\mathbf{F} \Rightarrow \mathbf{G}) \Rightarrow \mathbf{F} \Rightarrow \mathbf{H}$ (**Self-Distribution of Implication**),
(3) $(\neg\mathbf{F} \Rightarrow \neg\mathbf{G}) \Rightarrow \mathbf{G} \Rightarrow \mathbf{F}$ (**Contraposition**).

[4]Church [**8**, §10, p. 73] uses this term for this axiom. However, the term is also used for the fallacy of concluding $\mathbf{F}$ from $\mathbf{G}$ and $\mathbf{F} \Rightarrow \mathbf{G}$.

That is, the axioms are $\mathbf{K}_0(\mathbf{F},\mathbf{G})$, $\mathbf{K}_1(\mathbf{F},\mathbf{G},\mathbf{H})$, and $\mathbf{K}_2(\mathbf{F},\mathbf{G})$, where the formulas $\mathbf{K}_i$ are respectively $P_0 \Rightarrow P_1 \Rightarrow P_2$, $(P_0 \Rightarrow P_1 \Rightarrow P_2) \Rightarrow (P_0 \Rightarrow P_1) \Rightarrow P_0 \Rightarrow P_2$, and $(\neg P_0 \Rightarrow \neg P_1) \Rightarrow P_1 \Rightarrow P_0$. Our only rule of inference will be **Detachment,** or ***Modus Ponens,*** namely, that from formulas $\mathbf{F}$ and $\mathbf{F} \Rightarrow \mathbf{G}$, the formula $\mathbf{G}$ can be derived. If $\mathbf{F}$ is deducible from Σ, then we shall write

$$\Sigma \vdash \mathbf{F}.$$

If $\mathbf{F}$ is deducible from $\varnothing$, then we may just write

$$\vdash \mathbf{F}.$$

(See Appendix E on the origin of this notation.)

LEMMA 2.7.1. $\vdash \mathbf{F} \Rightarrow \mathbf{F}$.

PROOF. The following is a formal proof, with justifications.

(1)	$\mathbf{F} \Rightarrow (\mathbf{F} \Rightarrow \mathbf{F}) \Rightarrow \mathbf{F}$	[Affirmation of the Consequent]
(2)	$(\mathbf{F} \Rightarrow (\mathbf{F} \Rightarrow \mathbf{F}) \Rightarrow \mathbf{F}) \Rightarrow (\mathbf{F} \Rightarrow \mathbf{F} \Rightarrow \mathbf{F}) \Rightarrow \mathbf{F} \Rightarrow \mathbf{F}$	[Self-Distribution of $\Rightarrow$]
(3)	$(\mathbf{F} \Rightarrow \mathbf{F} \Rightarrow \mathbf{F}) \Rightarrow \mathbf{F} \Rightarrow \mathbf{F}$	[Detachment from (1) and (2)]
(4)	$\mathbf{F} \Rightarrow \mathbf{F} \Rightarrow \mathbf{F}$	[Affirmation of the Consequent]
(5)	$\mathbf{F} \Rightarrow \mathbf{F}$	[Detachment from (4) and (3)]

Thus $\vdash \mathbf{F} \Rightarrow \mathbf{F}$. □

The next two lemmas are immediate.

LEMMA 2.7.2. *If $\Sigma \vdash \mathbf{F}$, and $\Sigma \subseteq \Gamma$, then $\Gamma \vdash \mathbf{F}$.* □

LEMMA 2.7.3. *Every initial segment of a formal proof is itself a formal proof.* □

THEOREM 2.7.1. *The set of syntactic consequences of a set Σ is the set Γ given recursively by the following rules:*

(1) $\Sigma \subseteq \Gamma$;
(2) Γ contains the axioms;
(3) if $\mathbf{F} \in \Gamma$, and $\mathbf{F} \Rightarrow \mathbf{G}$ is in Γ, then $\mathbf{G} \in \Gamma$.

Thus the set of syntactic consequences of Σ admits induction.

PROOF. Let Γ' be the set of syntactic consequences of Σ. We first prove $\Gamma \subseteq \Gamma'$ by induction. If $\mathbf{F}$ belongs to Σ or is an axiom, then $\mathbf{F}$ is a one-line proof that $\Sigma \vdash \mathbf{F}$, so $\mathbf{F} \in \Gamma'$. If $\mathbf{F}$ and $\mathbf{F} \Rightarrow \mathbf{G}$ are in Γ', then they have formal proofs

$$(\mathbf{H}_1, \dots, \mathbf{H}_{m-1}, \mathbf{F}), \qquad (\mathbf{K}_1, \dots, \mathbf{K}_{n-1}, \mathbf{F} \Rightarrow \mathbf{G})$$

respectively; but then

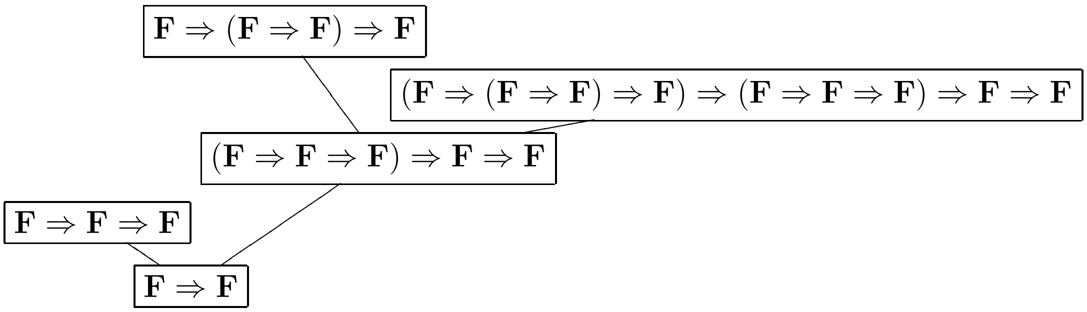
$$(\mathbf{H}_1, \dots, \mathbf{H}_{m-1}, \mathbf{F}, \mathbf{K}_1, \dots, \mathbf{K}_{n-1}, \mathbf{F} \Rightarrow \mathbf{G}, \mathbf{G})$$

is a formal proof of $\mathbf{G}$, so $\mathbf{G} \in \Gamma'$. By induction, $\Gamma \subseteq \Gamma'$.

Now we show $\Gamma' \subseteq \Gamma$ by induction on the lengths of formal proofs. Suppose $\mathbf{F} \in \Gamma$ whenever $\mathbf{F}$ has a formal proof from Σ of length less than n. Suppose $\mathbf{G}$ has a formal proof from Σ of length n. If $\mathbf{G}$ is an element of Σ or an axiom, then $\mathbf{G} \in \Gamma$ by definition. The only other possibility is that, in its proof, $\mathbf{G}$ is preceded by $\mathbf{F}$ and $(\mathbf{F} \Rightarrow \mathbf{G})$. Then, by inductive hypothesis and Lemma 2.7.3, both $\mathbf{F}$ and $\mathbf{F} \Rightarrow \mathbf{G}$ are in Γ, so that $\mathbf{G} \in \Gamma$. □

The proof that $\Gamma' \subseteq \Gamma$ is 'really' by induction on the *set* of formal proofs, when this is ordered so that the predecessors of a proof are its proper initial segments. Indeed, the set of formal proofs then becomes a tree, and trees admit proof by induction (exercise). In particular, one shows by induction that, for all formal proofs $(\mathbf{F}_0, \dots, \mathbf{F}_{n-1}, \mathbf{G})$ from Σ, we have $\mathbf{G} \in \Gamma$.

We can understand the last theorem as that formal proofs themselves correspond to certain trees. For example, the proof of $\mathbf{F} \Rightarrow \mathbf{F}$ can be written as the following tree.



However, the formula $\mathbf{F} \Rightarrow \mathbf{F}$ does not appear to carry within itself this tree: there is no obvious way to extract the tree from the formula.

Establishing syntactic entailment by formal proof is usually quite tedious. Theorem 2.7.1 allows some short-cuts, including the following.

COROLLARY 1.

(1) *if* $\mathbf{F} \in \Sigma$, *then* $\Sigma \vdash \mathbf{F}$*;*
(2) *if* $\mathbf{F}$ *is an axiom, then* $\Sigma \vdash \mathbf{F}$*;*
(3) ***Detachment:*** *if* $\Sigma \vdash \mathbf{F}$, *and* $\Sigma \vdash \mathbf{F} \Rightarrow \mathbf{G}$, *then* $\Sigma \vdash \mathbf{G}$. □

COROLLARY 2.

(1) ***Affirmation of the Consequent:*** *If* $\Sigma \vdash \mathbf{F}$, *then* $\Sigma \vdash \mathbf{G} \Rightarrow \mathbf{F}$.
(2) ***Self-Distribution of Implication:*** *If* $\Sigma \vdash \mathbf{F} \Rightarrow \mathbf{G}$ *and* $\Sigma \vdash \mathbf{F} \Rightarrow \mathbf{G} \Rightarrow \mathbf{H}$, *then* $\Sigma \vdash \mathbf{F} \Rightarrow \mathbf{H}$.
(3) ***Contraposition:*** *If* $\Sigma \vdash \neg\mathbf{F} \Rightarrow \neg\mathbf{G}$, *then* $\Sigma \vdash \mathbf{G} \Rightarrow \mathbf{F}$. □

More short-cuts are as follows.

THEOREM 2.7.2 (Deduction). $\Sigma \vdash \mathbf{F} \Rightarrow \mathbf{G} \iff \Sigma \cup \{\mathbf{F}\} \vdash \mathbf{G}$.

PROOF. The forward implication is an exercise. The reverse implication is by induction on the lengths of formal proofs. Suppose this implication holds for all $\mathbf{G}$ that have formal proofs shorter than the proof of $\mathbf{H}$; and suppose $\Sigma \cup \{\mathbf{F}\} \vdash \mathbf{H}$. With respect to the formal proof, there are three possibilities for $\mathbf{H}$.

If $\mathbf{H}$ is an axiom, or is one of the formulas in Σ, then $\Sigma \vdash \mathbf{H}$; hence $\Sigma \vdash \mathbf{F} \Rightarrow \mathbf{H}$ by Affirmation of the Consequent.

If $\mathbf{H}$ is $\mathbf{F}$, then $\vdash \mathbf{F} \Rightarrow \mathbf{H}$ by Lemma 2.7.1, so $\Sigma \vdash \mathbf{F} \Rightarrow \mathbf{H}$ by Lemma 2.7.2.

The last possibility is that, in its formal proof, $\mathbf{H}$ is preceded by some formulas $\mathbf{K}$ and $\mathbf{K} \Rightarrow \mathbf{H}$. By Lemma 2.7.3, these formulas are deducible from $\Sigma \cup \{\mathbf{F}\}$, and the inductive hypothesis applies to them. Therefore $\Sigma \vdash \mathbf{F} \Rightarrow \mathbf{K}$ and $\Sigma \vdash \mathbf{F} \Rightarrow \mathbf{K} \Rightarrow \mathbf{H}$. By Self-Distribution of Implication, $\Sigma \vdash \mathbf{F} \Rightarrow \mathbf{H}$. This completes the induction and the proof. □

Again, the proof is 'really' by induction in the tree of formal proofs.

LEMMA 2.7.4. *If $\Sigma \vdash \mathbf{F}$, and $\Gamma \vdash \mathbf{G}$ for all $\mathbf{G}$ in Σ, then $\Gamma \vdash \mathbf{F}$.* □

LEMMA 2.7.5. *The following are deducible from $\varnothing$:*

(1) $\neg\mathbf{G} \Rightarrow \mathbf{G} \Rightarrow \mathbf{F}$*;*
(2) $\neg\neg\mathbf{F} \Rightarrow \mathbf{F}$*;*
(3) $\mathbf{F} \Rightarrow \neg\neg\mathbf{F}$*;*
(4) $(\mathbf{F} \Rightarrow \mathbf{G}) \Rightarrow \neg\mathbf{G} \Rightarrow \neg\mathbf{F}$*;*
(5) $\mathbf{F} \Rightarrow \neg\mathbf{G} \Rightarrow \neg(\mathbf{F} \Rightarrow \mathbf{G})$*.*
(6) $(\mathbf{F} \Rightarrow \mathbf{G}) \Rightarrow (\neg\mathbf{F} \Rightarrow \mathbf{G}) \Rightarrow \mathbf{G}$*.*

PROOF. We have

$$\begin{aligned}
&\{\neg\mathbf{G}\} \vdash \neg\mathbf{G}, && \\
&\{\neg\mathbf{G}\} \vdash \neg\mathbf{F} \Rightarrow \neg\mathbf{G}, && \text{[Affirmation of the Consequent]}\\
&\{\neg\mathbf{G}\} \vdash \mathbf{G} \Rightarrow \mathbf{F}, && \text{[Contraposition]}\\
&\vdash \neg\mathbf{G} \Rightarrow \mathbf{G} \Rightarrow \mathbf{F}, && \text{[Deduction]}
\end{aligned}$$

and thus (1). As a special case of the penultimate conclusion,

$$\begin{aligned}
&\{\neg\neg\mathbf{F}\} \vdash \neg\mathbf{F} \Rightarrow \neg\neg\neg\mathbf{F}, && \\
&\{\neg\neg\mathbf{F}\} \vdash \neg\neg\mathbf{F} \Rightarrow \mathbf{F}, && \text{[Contraposition]}\\
&\{\neg\neg\mathbf{F}\} \vdash \mathbf{F}, && \text{[Deduction]}\\
&\vdash \neg\neg\mathbf{F} \Rightarrow \mathbf{F}, && \text{[Deduction]}
\end{aligned}$$

so (2). Part (3) is an exercise. For (4), we have

$$\begin{aligned}
&\{\mathbf{F} \Rightarrow \mathbf{G}, \mathbf{F}\} \vdash \mathbf{G}, && \\
&\{\mathbf{F} \Rightarrow \mathbf{G}, \neg\neg\mathbf{F}\} \vdash \mathbf{G}, && \text{[Lemma 2.7.4 and (2)]}\\
&\{\mathbf{F} \Rightarrow \mathbf{G}, \neg\neg\mathbf{F}\} \vdash \neg\neg\mathbf{G}, && \text{[(3)]}\\
&\{\mathbf{F} \Rightarrow \mathbf{G}\} \vdash \neg\neg\mathbf{F} \Rightarrow \neg\neg\mathbf{G}, && \\
&\{\mathbf{F} \Rightarrow \mathbf{G}\} \vdash \neg\mathbf{G} \Rightarrow \neg\mathbf{F}, && \\
&\vdash (\mathbf{F} \Rightarrow \mathbf{G}) \Rightarrow \neg\mathbf{G} \Rightarrow \neg\mathbf{F}. &&
\end{aligned}$$

The remaining (5) and (6) are an exercise. □

2.8. Completeness

An arbitrary proof system is

(1) **sound,** if every set of formulas logically entails its syntactic consequences;
(2) **complete,** if every set of formulas syntactically entails its logical consequences.

We shall show that our proof system is sound and complete.

THEOREM 2.8.1 (Soundness). *If $\Sigma \vdash \mathbf{F}$, then $\mathbf{F} \in \mathrm{Con}(\Sigma)$.*

PROOF. We use induction on the set of syntactic consequences of Σ (that is, Theorem 2.7.1) to show that it is a subset of $\mathrm{Con}(\Sigma)$. All elements of Σ are logical consequences of Σ. Since all axioms are tautologies, they are logical consequences of Σ. Finally, suppose $\mathbf{F}$ and $\mathbf{F} \Rightarrow \mathbf{G}$ are logical consequences of Σ, and ε is a model of Σ. Then $\widehat{\mathbf{F}}(\varepsilon) = 1$. Also, writing $\mathbf{H}$ for $\mathbf{F} \Rightarrow \mathbf{G}$, we have

$$1 = \widehat{\mathbf{H}}(\varepsilon) = 1 + \widehat{\mathbf{F}}(\varepsilon) + \widehat{\mathbf{F}}(\varepsilon) \cdot \widehat{\mathbf{G}}(\varepsilon) = 1 + 1 + 1 \cdot \widehat{\mathbf{G}}(\varepsilon) = \widehat{\mathbf{G}}(\varepsilon),$$

so $\varepsilon \vDash \mathbf{G}$. This completes the induction and the proof. □

Proving completeness will take more work.[5]

LEMMA 2.8.1. *Let $\vec{e}$ be an n-ary truth-assignment, and for all n-ary formulas $\mathbf{F}$, let*

$$\mathbf{F}' = \begin{cases} \mathbf{F}, & \text{if } \widehat{\mathbf{F}}(\vec{e}) = 1; \\ \neg\mathbf{F}, & \text{if } \widehat{\mathbf{F}}(\vec{e}) = 0. \end{cases}$$

Then

$$\{P_0{}', \dots, P_{n-1}{}'\} \vdash \mathbf{F}'.$$

PROOF. We use induction on n-ary formulas. Let $\Sigma = \{P_0{}', \dots, P_{n-1}{}'\}$. If $\mathbf{F}$ is a variable P_k, where $k < n$, then $\mathbf{F}'$ is in Σ, so $\Sigma \vdash \mathbf{F}'$. Suppose the claim holds when $\mathbf{F}$ is $\mathbf{G}$. Let $\mathbf{F}$ be $\neg\mathbf{G}$. There are two cases to consider.

(1) If $\widehat{\mathbf{F}}(\vec{e}) = 1$, then $\widehat{\mathbf{G}}(\vec{e}) = 0$, so $\mathbf{F}'$ is $\mathbf{F}$, but $\mathbf{G}'$ is $\neg\mathbf{G}$, which is $\mathbf{F}$, that is, $\mathbf{F}'$.
(2) If $\widehat{\mathbf{F}}(\vec{e}) = 0$, then $\widehat{\mathbf{G}}(\vec{e}) = 1$, so $\mathbf{G}'$ is $\mathbf{G}$, but $\mathbf{F}'$ is $\neg\mathbf{F}$, which is $\neg\neg\mathbf{G}$, that is, $\neg\neg\mathbf{G}'$.

Since we assume $\Sigma \vdash \mathbf{G}'$, we immediately have $\Sigma \vdash \mathbf{F}'$ in the first case. In the second case, we have $\Sigma \vdash \mathbf{G}$, hence $\Sigma \vdash \neg\neg\mathbf{G}$, that is, $\Sigma \vdash \mathbf{F}'$, by Lemma 2.7.5(3) and Detachment. Suppose finally that the claim holds when $\mathbf{F}$ is $\mathbf{G}$ or $\mathbf{H}$. Let $\mathbf{F}$ be $\mathbf{G} \Rightarrow \mathbf{H}$. There are three cases to consider:

(1) $\widehat{\mathbf{G}}(\vec{e}) = 0$;
(2) $\widehat{\mathbf{H}}(\vec{e}) = 1$;
(3) $\widehat{\mathbf{G}}(\vec{e}) = 1$ and $\widehat{\mathbf{H}}(\varepsilon) = 0$.

Details are left to the reader. This completes the proof. □

THEOREM 2.8.2 (Completeness). *If* $\mathbf{F} \in \mathrm{Con}(\Sigma)$*, then* $\Sigma \vdash \mathbf{F}$.

PROOF. Suppose $\mathbf{F} \in \mathrm{Con}(\Sigma)$. By Compactness (rather, its corollary), Σ has a finite subset Γ such that $\mathbf{F} \in \mathrm{Con}(\Gamma)$. Write Γ as $\{\mathbf{F}_0, \dots, \mathbf{F}_{m-1}\}$, and $\mathbf{F}$ as $\mathbf{F}_m$. Then the formula

$$\mathbf{F}_0 \Rightarrow \cdots \Rightarrow \mathbf{F}_m$$

is a tautology (exercise). Call this tautology $\mathbf{G}$, and suppose it is n-ary. Let $P_k{}' \in \{P_k, \neg P_k\}$ for each k in n. By the previous lemma, we have

$$\{P_0{}' \dots, P_{n-1}{}'\} \vdash \mathbf{G}. \tag{$*$}$$

By the Deduction Theorem (2.7.2), remembering that $P_\ell{}'$ can be either P_ℓ or $\neg P_\ell$, we have

$$\{P_0{}' \dots, P_{n-2}{}'\} \vdash P_{n-1} \Rightarrow \mathbf{G},$$
$$\{P_0{}' \dots, P_{n-2}{}'\} \vdash \neg P_{n-1} \Rightarrow \mathbf{G}.$$

so $\{P_0{}' \dots, P_{n-2}{}'\} \vdash \mathbf{G}$ by Lemma 2.7.5(6). Continuing this elimination process, we arrive at the conclusion $\vdash \mathbf{G}$, that is, $\vdash \mathbf{F}_0 \Rightarrow \cdots \Rightarrow \mathbf{F}_{m-1} \Rightarrow \mathbf{F}$. By Deduction in the other direction, $\{\mathbf{F}_0, \cdots, \mathbf{F}_{m-1}\} \vdash \mathbf{F}$. □

[5]The following lemma corresponds to one found in Church [**8**, *151, p. 98]; the origin is not clear.

2.9. Other propositional logics

An arbitrary signature $\mathcal{L}$ for propositional logic may have connectives of any arity. Then the formulas in $\mathcal{L}$ can be written in Polish notation, as terms are in §1.8, so that

(1) each variable is a formula;
(2) $*\mathbf{F}_0\cdots\mathbf{F}_{n-1}$ is a formula, if $*$ is an n-ary connective from $\mathcal{L}$, and the $\mathbf{F}_i$ are formulas. (If $n=0$, then $*$ by itself is a formula.)

Then the set of formulas admits recursion by Theorem 1.8.3.

Each connective is given an interpretation as an operation on $\mathbb{B}$; from these, and a truth-assignment ε, a function $\mathbf{F}\mapsto\widehat{\mathbf{F}}(\varepsilon)$ is determined as in §2.4. We may say then that an n-ary formula $\mathbf{F}$ **represents** the n-ary operation $\vec{e}\mapsto\widehat{\mathbf{F}}(\vec{e})$ or $\widehat{\mathbf{F}}$ on $\mathbb{B}$. The formal definition is recursive:

(1) If $k<n$, then the formula P_k is an n-ary formula and, as such, represents the operation $\vec{e}\mapsto e_k$.
(2) If $(\mathbf{F}_0,\dots,\mathbf{F}_{k-1})$ is an k-tuple of formulas, each of them n-ary, and if $*$ is a k-ary connective in $\mathcal{L}$, then the formula $*\mathbf{F}_0\cdots\mathbf{F}_{k-1}$ represents the function
$$\vec{x}\longmapsto g(\widehat{\mathbf{F}}_0(\vec{x}\,),\dots,\widehat{\mathbf{F}}_{k-1}(\vec{x}\,))$$
from $\mathbb{B}^n$ to $\mathbb{B}$, where g is the standard interpretation of $*$.

In particular, if $*$ is n-ary, then its standard interpretation is $\widehat{\mathbf{G}}$, where $\mathbf{G}$ is the formula $*P_0\cdots P_{n-1}$.

Each n-ary operation g on $\mathbb{B}$ determines, for each k, the $(n+k)$-ary operation
$$(\vec{x}\,,\vec{y}\,)\longmapsto g(\vec{x}\,).$$

If $\mathbf{F}$ is an $(n+k)$-ary formula representing this operation, let us say also that $\mathbf{F}$ represents g itself. Then a signature $\mathcal{L}$ for a propositional logic is **adequate** if each operation on $\mathbb{B}$ is represented by a formula of the logic. The following basic tool for establishing adequacy of a signature was proved by Emil Post in 1921 [**28**]:

LEMMA 2.9.1. *A signature of propositional logic is adequate, provided that, in this signature, the following operations are represented:*

(1) the nullary operations 0 *and* 1*;*
(2) the ternary operation f *given by the following table.*

e_0	e_1	e_2	$f(\vec{e})$
0	0	0	0
1	0	0	1
0	1	0	0
1	1	0	1
0	0	1	0
1	0	1	0
0	1	1	1
1	1	1	1

PROOF. We use induction on the arity of operations. The nullary operations are represented in the signature by assumption. Suppose all n-ary operations are represented, and g is $(n+1)$-ary. If $e \in \mathbb{B}$, let h_e be the n-ary operation $\vec{x} \mapsto g(\vec{x}, e)$. By definition,

$$f(e_0, e_1, e_2) = \begin{cases} e_0, & \text{if } e_2 = 0; \\ e_1, & \text{if } e_2 = 1. \end{cases}$$

Then for all $\vec{d}$ in $\mathbb{B}^n$, we have

$$g(\vec{d}, e) = h_e(\vec{d}) = f(h_0(\vec{d}), h_1(\vec{d}), e).$$

Thus the operation g is

$$(\vec{x}, y) \longmapsto f(h_0(\vec{x}), h_1(\vec{x}), y).$$

By inductive hypothesis, each of the operations h_e is represented by some formula

$$\mathbf{H}_e(P_0, \dots, P_{n-1}, \dots);$$

by assumption, f is represented by some formula $\mathbf{F}(P_0, P_1, P_2, \dots)$. Hence g is represented by

$$\mathbf{F}(\mathbf{H}_0(P_0, \dots, P_{n-1}, \dots), \mathbf{H}_1(P_0, \dots, P_{n-1}, \dots), P_n, \dots).$$

By induction, the operations of all arities are represented. □

THEOREM 2.9.1. *The propositional signature* $\{\neg, \Rightarrow\}$ *is adequate.*

PROOF. By the lemma, it is enough to observe that $P_0 \Rightarrow P_0$ represents 1, and $\neg(P_0 \Rightarrow P_0)$ represents 0, while the formula $\neg((\neg P_2 \Rightarrow P_0) \Rightarrow \neg(P_2 \Rightarrow P_1))$ has the truth-table

$\neg$	$((\neg$	P_2	$\Rightarrow$	$P_0)$	$\Rightarrow$	$\neg$	$(P_2$	$\Rightarrow$	$P_1))$
0	1	0	0	0	1	0	0	1	0
1	1	0	1	1	0	0	0	1	0
0	1	0	0	0	1	0	0	1	1
1	1	0	1	1	0	0	0	1	1
0	0	1	1	0	1	1	1	0	0
0	0	1	1	1	1	1	1	0	0
1	0	1	1	0	0	0	1	1	1
1	0	1	1	1	0	0	1	1	1

and so represents the operation in the lemma. □

The propositional signature $\{\neg, \Rightarrow\}$ may be *adequate,* but it is not very *useful* for doing mathematics. It is perhaps more convenient to use the propositional signature

$$\{\neg, \Rightarrow, \&, \vee, \Leftrightarrow\}$$

and use **infix notation** for formulas, as we did for PF, so that

(1) variables are formulas;
(2) if $\mathbf{A}$ is a formula, then so is $\neg\mathbf{A}$;
(3) if $\mathbf{A}$ and $\mathbf{B}$ are formulas, then so is $(\mathbf{A} * \mathbf{B})$, where $*$ is $\Rightarrow$, $\&$, $\vee$, or $\Leftrightarrow$.

The interpretations of the new connectives were given in §1.9. Note then that the formula in the last theorem is equivalent to $((\neg P_2 \Rightarrow P_0) \;\&\; (P_2 \Rightarrow P_1))$. In writing formulas, we may follow the conventions established in §2.3 for PF′, so that, for example, we may omit the outer parentheses. We may also remove interior parentheses, with the understanding that $\&$ and $\vee$ are more binding then $\Rightarrow$ and $\Leftrightarrow$. Then, for example, instead of $((\mathbf{F} \;\&\; \mathbf{G}) \Rightarrow \mathbf{H})$, we may just write $\mathbf{F} \;\&\; \mathbf{G} \Rightarrow \mathbf{H}$. Since the interpretation

& (namely, multiplication on $\mathbb{B}$) is associative, the interpretation of $\mathbf{F} \mathbin{\&} \mathbf{G} \mathbin{\&} \mathbf{H}$ is unambiguous; likewise for $\mathbf{F} \vee \mathbf{G} \vee \mathbf{H}$.

For the purposes of writing recursive definitions and inductive proofs, it will be convenient to think of our official signature as $\{\neg, \&\}$. We can do this, because

COROLLARY. *The propositional signature* $\{\neg, \&\}$ *is adequate.*

PROOF. $(\mathbf{F} \Rightarrow \mathbf{G})$ is equivalent to $\neg(\mathbf{F} \mathbin{\&} \neg\mathbf{G})$. □

Exercises

EXERCISE 2.1. Prove by induction that

(1) every propositional formula has the same number of left as right parentheses;
(2) an entry $\neg$ is never preceded by a variable in any formula.

EXERCISE 2.2. As in the example on p. 27, draw trees for some formulas, such as

(1) $(P_0 \Rightarrow (P_1 \Rightarrow P_0))$,
(2) $((P_0 \Rightarrow (P_1 \Rightarrow P_2)) \Rightarrow ((P_0 \Rightarrow P_1) \Rightarrow (P_0 \Rightarrow P_2)))$,
(3) $((\neg P_0 \Rightarrow \neg P_1) \Rightarrow (P_1 \Rightarrow P_0))$.

EXERCISE 2.3. Supply the missing details in the proof of the Recursion Theorem (2.2.1) and its corollary.

EXERCISE 2.4. Prove the Recursion Theorem for Formulas under the assumption that all formulas are written in Łukasiewicz notation (see §2.3).

EXERCISE 2.5. Formulate and prove an analogue of Theorem 2.2.1 for PF'.

EXERCISE 2.6. Complete the proof of Theorem 2.3.1.

EXERCISE 2.7. Construct truth-tables for some formulas, such as those in Exercise 2.2.

EXERCISE 2.8.

(1) Complete the proof of Theorem 2.4.1.
(2) Prove Lemma 2.4.1.
(3) Prove the Replacement Theorem (2.4.3).

EXERCISE 2.9. Let Σ be $\{P_0 \Rightarrow P_1, P_1 \Rightarrow P_2\}$.

(1) Find $\mathrm{Mod}(\Sigma)$.
(2) Find a formula $\mathbf{F}$ such that
$$\mathrm{Mod}(\Sigma) = \mathrm{Mod}(\{\mathbf{F}\}).$$
(3) Find a formula $\mathbf{G}$ such that $\mathbf{G} \in \mathrm{Con}(\Sigma)$ but $\mathrm{Mod}(\{\mathbf{G}\}) \neq \mathrm{Mod}(\Sigma)$.

EXERCISE 2.10. Prove Lemma 2.5.2.

EXERCISE 2.11. Prove that Σ is a theory if and only if $\mathrm{Con}(\Sigma) = \Sigma$.

EXERCISE 2.12. Can you find a formula $\mathbf{F}$ such that $\mathrm{Con}(\{\mathbf{F}\})$ is

(1) PF?
(2) $\varnothing$?

EXERCISE 2.13.
(1) If $\mathrm{Con}(\{\mathbf{F}\}) \subseteq \mathrm{Con}(\{\mathbf{G}\})$, must $\mathbf{G}$ logically entail $\mathbf{F}$?
(2) How is $\mathrm{Con}(\{\mathbf{F} \Rightarrow \mathbf{G}\})$ related to $\mathrm{Con}(\{\neg\mathbf{F}\})$ and $\mathrm{Con}(\{\mathbf{G}\})$?
(3) Can you find pairwise-inequivalent formulas $\mathbf{F}$, $\mathbf{G}$, and $\mathbf{H}$ such that

$$\mathrm{Con}(\{\mathbf{F}\}) \cup \mathrm{Con}(\{\mathbf{G}\}) = \mathrm{Con}(\{\mathbf{H}\})?$$

(4) If Σ logically entails $\mathbf{F} \Rightarrow \mathbf{G}$, must it entail either $\neg\mathbf{F}$ or $\mathbf{G}$?

EXERCISE 2.14.
(1) Show that

$$\bigcap_{i\in I} \mathrm{Mod}(\Sigma_i) = \mathrm{Mod}(\bigcup_{i\in I} \Sigma_i).$$

(2) Show that $\mathrm{Mod}(\{\mathbf{F}\}) \cup \mathrm{Mod}(\{\mathbf{G}\}) = \mathrm{Mod}(\{\neg\mathbf{F} \Rightarrow \mathbf{G}\})$.

(This shows that the sets $\mathrm{Mod}(\Sigma)$ are the *closed* sets in a topology for $\mathbb{B}^V$. Then the Compactness Theorem is that this topology is compact.)

EXERCISE 2.15. Show that $\mathrm{Mod}(\{\mathbf{F}\})^{\mathrm{c}} = \mathrm{Mod}(\{\neg\mathbf{F}\})$.

EXERCISE 2.16. If $\Gamma \cup \{\mathbf{F}\}$ is not satisfiable, why is $\Gamma \cup \{\neg\neg\mathbf{F}\}$ not satisfiable?

EXERCISE 2.17. Why has the set $\{\mathbf{F}, \neg\mathbf{F}\}$ no models?

EXERCISE 2.18. In the proof of the Compactness Theorem, why does ζ agree with ε on $\{P_k \colon k < n\}$?

EXERCISE 2.19. In the example in §2.6:
(1) Give a precise recursive definition of the sets Σ_n.
(2) Prove that $\varepsilon_n \in \mathrm{Mod}(\Sigma_n) \smallsetminus \mathrm{Mod}(\Sigma_{n+1})$.
(3) What is $\mathrm{Con}(\Sigma)$?

EXERCISE 2.20. Suppose I is a set, and there is a function $i \mapsto \mathbf{F}_i$ from I into PF, such that

$$\bigcup_{i\in I} \mathrm{Mod}(\{\mathbf{F}_i\}) = \mathbb{B}^V.$$

Prove that I has a finite subset J such that $\bigcup_{i\in J} \mathrm{Mod}(\{\mathbf{F}_i\}) = \mathbb{B}^V$.

EXERCISE 2.21. Define a binary operation $*$ on PF such that, for each formula $\mathbf{F}$, the function

$$\mathbf{G} \mapsto \mathbf{F} * \mathbf{G}$$

is recursively defined, and $\mathbf{F} * (\mathbf{F} \Rightarrow \mathbf{G}) = \mathbf{G}$.

EXERCISE 2.22. Prove that all trees admit proof by induction.

EXERCISE 2.23. Prove the corollaries of Theorem 2.7.1.

EXERCISE 2.24. Prove the forward implication of the Deduction Theorem.

EXERCISE 2.25. Prove Lemma 2.7.4.

EXERCISE 2.26. Prove the remainder of Lemma 2.7.5.

EXERCISE 2.27. Complete the proof of Lemma 2.8.1.

EXERCISE 2.28. Prove that, if $\mathbf{F}_m \in \mathrm{Con}(\{\mathbf{F}_0, \ldots, \mathbf{F}_{m-1}\})$, then the formula

$$\mathbf{F}_0 \Rightarrow \cdots \Rightarrow \mathbf{F}_m$$

is a tautology.

EXERCISE 2.29. Prove that $\{|\}$ is adequate, where $|$ (the *Sheffer stroke*) is given by

P	$\mid$	Q
0	1	0
1	1	0
0	1	1
1	0	1

EXERCISE 2.30. Prove that $\{\&, \vee\}$ is not adequate.

CHAPTER 3

First-order logic

Throughout this chapter, let $\mathfrak{A}$ stand for an arbitrary structure (in the sense of §1.7); its signature will be $\mathcal{L}$. So $\mathfrak{A}$ has universe A, which is just a set. We shall use the letters c, R and f to stand for arbitrary constants, predicates and function-symbols of $\mathcal{L}$, respectively. The arity of R and and of f will be n.

Instead of propositional variables, we shall use the set of **individual variables,**

$$\{\mathsf{x}_k \colon k \in \omega\},$$

introduced in §1.8. The definition of **terms** there makes sense for arbitrary structures. Much of the account of terms given below will be a retelling of the account in §1.8. Terms do not involve predicates, but symbolize the ways of combining basic operations to get new operations. The ways of combining these with basic relations to get new relations will be symbolized by *(first-order) formulas.*

Our logic will be **first order,** because we shall use only individual variables, and not *set variables.* This means that the notions of induction and recursion cannot be formulated in first-order logic. This deficiency is compensated for by the Compactness Theorem, 6.1.1 below, which fails in second-order logic.

3.1. Terms

If $k < n$, then there is an n-ary operation

$$\vec{x} \longmapsto x_k \tag{$*$}$$

on A. This operation is **projection** onto the kth coordinate. Each element b of A determines, for each positive n, the constant n-ary operation

$$\vec{x} \longmapsto b. \tag{$\dagger$}$$

If b is $c^{\mathfrak{A}}$, then we have the n-ary operation $\vec{x} \mapsto c^{\mathfrak{A}}$. More generally, if α is an n-ary operation on A, then there is an $(n+k)$-ary operation on A, namely

$$(\vec{x}\,, \vec{y}\,) \longmapsto \alpha(\vec{x}\,).$$

All operations on A that are symbolized in $\mathcal{L}$ can be composed with one another and with projections to give other operations on A. The **terms** of $\mathcal{L}$ symbolize these new operations. The symbols used in terms of $\mathcal{L}$ are:

(1) the variables x_i, which will symbolize the projections;
(2) the constants c of $\mathcal{L}$;
(3) the function-symbols f of $\mathcal{L}$.

Then the terms of $\mathcal{L}$ are defined inductively thus:

(1) Each individual variable is a term of $\mathcal{L}$.
(2) Each constant in $\mathcal{L}$ is a term of $\mathcal{L}$.

(3) If f is an n-ary function-symbol of $\mathcal{L}$, and $t_0, \dots, t_{n-1}$ are terms of $\mathcal{L}$, then the string
$$ft_0 \cdots t_{n-1}$$
is a term of $\mathcal{L}$.

(Note well that $ft_0 \cdots t_{n-1}$ is not generally a string of length $n+1$. If f is binary, then we may unofficially write the term as $(t_0\ f\ t_1)$ instead of ft_0t_1.) Let the set of terms of $\mathcal{L}$ be denoted by
$$\mathrm{Tm}(\mathcal{L}).$$
This set admits recursion by Theorem 1.8.3. If the variables in a term t come from $\{\mathsf{x}_k \colon k < n\}$, then t is n**-ary;** the set of n-ary terms of $\mathcal{L}$ can be denoted by
$$\mathrm{Tm}^n(\mathcal{L}).$$
Note then
$$\mathrm{Tm}^0(\mathcal{L}) \subseteq \mathrm{Tm}^1(\mathcal{L}) \subseteq \mathrm{Tm}^2(\mathcal{L}) \subseteq \cdots .$$
The nullary terms are the **constant terms.** By the recursive definition in §1.8, a constant term is interpreted in $\mathfrak{A}$ as an element of A; an n-ary term, as an n-ary operation on A:

(1) ${\mathsf{x}_k}^{\mathfrak{A}}$ is $\vec{x} \mapsto x_k$ (as in $(*)$), if $k < n$.
(2) $c^{\mathfrak{A}}$ is $\vec{x} \mapsto c^{\mathfrak{A}}$ (as in (†); here c is understood respectively as term and constant).
(3) $(ft_0 \cdots t_{n-1})^{\mathfrak{A}}$ is
$$\vec{x} \longmapsto f^{\mathfrak{A}}({t_0}^{\mathfrak{A}}(\vec{x}\,), \dots, {t_{n-1}}^{\mathfrak{A}}(\vec{x}\,)),$$
that is, $f^{\mathfrak{A}} \circ ({t_0}^{\mathfrak{A}}, \dots, {t_{n-1}}^{\mathfrak{A}})$.

We may say that t **represents** the operation $t^{\mathfrak{A}}$ on A.

For example, say $\mathcal{L}$ is the signature $\{+, -, \cdot, 0, 1\}$ of fields, and $\mathfrak{A}$ is an infinite field (such as $\mathbb{Q}$ or $\mathbb{R}$ or $\mathbb{C}$). If t is a term of $\mathcal{L}(A)$, then $t^{\mathfrak{A}}$ is a **polynomial** over A. But a difficulty arises when $\mathfrak{A}$ is a finite field, such as $\mathbb{F}_2$. In this case, if t is either $\mathsf{x}_0 \cdot (\mathsf{x}_0 + 1)$ or 0, then $t^{\mathfrak{A}}(a) = 0$ for both a in A. However, the two terms may represent different polynomials over larger fields, such as $\mathbb{F}_4$ (which can be defined as $\mathbb{F}_2[X]/(X^2+1)$).

If t is an n-ary term, and $u_0, \dots, u_{n-1}$ are m-ary terms, then (as in §2.1), by substituting u_i for x_i, we obtain the m-ary term denoted by
$$t(u_0, \dots, u_{n-1}).$$
For example, if t is n-ary, then t is precisely the term denoted by
$$t(\mathsf{x}_0, \dots, \mathsf{x}_{n-1}).$$
We have a generalization of Theorem 2.4.1:

Theorem 3.1.1 (Associativity). *In a signature $\mathcal{L}$, if t is an n-ary term, and $u_0, \dots, u_{n-1}$ are m-ary terms, then*
$$t(u_0, \dots, u_{n-1})^{\mathfrak{A}} = t^{\mathfrak{A}} \circ ({u_0}^{\mathfrak{A}}, \dots, {u_{n-1}}^{\mathfrak{A}})$$
for all $\mathcal{L}$-structures $\mathfrak{A}$. □

An important special case arises as follows. Suppose $\mathcal{L} \subseteq \mathcal{L}'$. An **expansion** of $\mathfrak{A}$ to $\mathcal{L}'$ is a structure $\mathfrak{A}'$ whose signature is $\mathcal{L}'$, and whose universe is A, such that
$$s^{\mathfrak{A}'} = s^{\mathfrak{A}}$$
for all s in $\mathcal{L}$. Then $\mathfrak{A}$ is the **reduct** of $\mathfrak{A}'$ to $\mathcal{L}$. For example, the ring $(\mathbb{Z}, +, -, \cdot, 0, 1)$ is an expansion of $(\mathbb{Z}, +, -, 0)$, an abelian group; the latter is a reduct of the former.

We can treat the elements of A as new constants[1] (not belonging to $\mathcal{L}$); adding these to $\mathcal{L}$ gives the signature $\mathcal{L}(A)$. Then $\mathfrak{A}$ has a natural expansion $\mathfrak{A}_A$ to this signature, so that

$$a^{\mathfrak{A}_A} = a$$

for all a in A. We shall also want to speak of expansions $\mathfrak{A}_X$ of $\mathfrak{A}$, where X is an arbitrary subset of A.

An n-tuple $\vec{a}$ from $\mathfrak{A}$ determines a function $t \mapsto t(\vec{a}\,)$ from $\mathrm{Tm}^n(\mathcal{L})$ to $\mathrm{Tm}^0(\mathcal{L}(A))$. The tuple $\vec{a}$ also determines the function $g \mapsto g(\vec{a}\,)$ from A^{A^n} to A. Then we have two functions, $t \mapsto t^{\mathfrak{A}}(\vec{a}\,)$ and $t \mapsto t(\vec{a}\,)^{\mathfrak{A}}$, from $\mathrm{Tm}^n(\mathcal{L})$ into A. These can be understood as two paths in the following diagram.

$$\begin{tikzcd}
\mathrm{Tm}^n(\mathcal{L}) \arrow[r, "\vec{a}"] \arrow[d, "\mathfrak{I}"'] & \mathrm{Tm}^0(\mathcal{L}(A)) \arrow[d, "\mathfrak{I}"] \\
A^{A^n} \arrow[r, "\vec{a}"'] & A
\end{tikzcd}$$

By Associativity, it doesn't matter which way one moves around this diagram:

$$t^{\mathfrak{A}}(\vec{a}\,) = t(\vec{a}\,)^{\mathfrak{A}_A}. \tag{‡}$$

In a word, the diagram **commutes.**

3.2. Atomic formulas

As terms symbolize operations, so *formulas* will symbolize relations. Each n-ary formula φ of $\mathcal{L}$ will, for each $\mathcal{L}$-structure $\mathfrak{A}$, have an *interpretation* $\varphi^{\mathfrak{A}}$ as an n-ary relation on A. A nullary formula will be a **sentence.** Hence, if σ is a sentence of $\mathcal{L}$, then $\sigma^{\mathfrak{A}} \in \mathbb{B}$. If $\sigma^{\mathfrak{A}} = 1$, then σ is **true in** $\mathfrak{A}$, and we write

$$\mathfrak{A} \models \sigma.$$

In practice, it will be easier to define truth *before* defining interpretations in general.

So-called polynomial equations are examples of *atomic formulas,* which are the first kinds of formulas to be defined. From these, we shall define *open formulas,* and then arbitrary *formulas.*

The **atomic formulas** of $\mathcal{L}$ are of two kinds:

(1) If t_0 and t_1 are terms of $\mathcal{L}$, then the **equation** $t_0 = t_1$ is an atomic formula of $\mathcal{L}$.[2]
(2) If R is an n-ary predicate of $\mathcal{L}$, and $t_0, \dots, t_{n-1}$ are terms of $\mathcal{L}$, then $Rt_0 \cdots t_{n-1}$ is an atomic formula of $\mathcal{L}$. (If R is binary, then we may unofficially write $(t_0\ R\ t_1)$ instead of Rt_0t_1.)

An atomic formula α can be called k**-ary** if the terms it is made from are k-ary.

A polynomial equation over $\mathbb{R}$ has a solution-set, which can be considered as the *interpretation* of the equation in $\mathbb{R}$. Likewise, arbitrary atomic formulas have solution-sets, which are their interpretations: If α is a k-ary atomic formula of $\mathcal{L}$, then the

[1]If $a \in A$, some writers prefer denote by c_a the new constant whose interpretation in $\mathfrak{A}$ is a.
[2]Some writers prefer to use a symbol like $\equiv$ instead of $=$.

interpretation in $\mathfrak{A}$ of α is the k-ary relation $\alpha^{\mathfrak{A}}$ on A defined as follows. (Strictly, the validity of the definition depends on unique readability, given by Theorem 3.4.1 below.)

$$(t_0 = t_1)^{\mathfrak{A}} = \{\vec{x} \in A^k \colon {t_0}^{\mathfrak{A}}(\vec{x}\,) = {t_1}^{\mathfrak{A}}(\vec{x}\,)\};$$
$$(Rt_0 \cdots R_{n-1})^{\mathfrak{A}} = \{\vec{x} \in A^k \colon ({t_0}^{\mathfrak{A}}(\vec{x}\,), \dots, {t_{n-1}}^{\mathfrak{A}}(\vec{x}\,)) \in R^{\mathfrak{A}}\}.$$

As a special case, if $k = 0$, we have

$$\begin{aligned} (t_0 = t_1)^{\mathfrak{A}} = 1 &\iff {t_0}^{\mathfrak{A}} = {t_1}^{\mathfrak{A}}; \\ (Rt_0 \cdots t_{n-1})^{\mathfrak{A}} = 1 &\iff ({t_0}^{\mathfrak{A}}, \dots, {t_{n-1}}^{\mathfrak{A}}) \in R^{\mathfrak{A}}. \end{aligned} \tag{$*$}$$

Note that the atomic formula $t_0 = t_1$ can be considered as the special case of $Rt_0 \cdots t_{n-1}$ when $n = 2$ and R is $=$. We treat the special case separately because we consider the equals-sign to be *always* available for use in formulas, and we *always* interpret it as equality.

3.3. Open formulas

An **open** or **quantifier-free formula** is obtained from a propositional formula by substituting atomic formulas for the propositional variables. Let $\mathbf{F}$ be an n-ary propositional formula, and let σ_0, …, σ_{n-1} be atomic *sentences.* Then the **interpretation** of the open sentence $\mathbf{F}(\sigma_0, \dots, \sigma_{n-1})$ in $\mathfrak{A}$ is given by

$$\mathbf{F}(\sigma_0, \dots, \sigma_{n-1})^{\mathfrak{A}} = \widehat{\mathbf{F}}({\sigma_0}^{\mathfrak{A}}, \dots, {\sigma_{n-1}}^{\mathfrak{A}}).$$

Now suppose more generally that φ_0, …, φ_{n-1} are m-ary atomic formulas. Let θ be the m-ary open formula $\mathbf{F}(\varphi_0, \dots, \varphi_{n-1})$. If $\vec{a} \in A^m$, then $\theta(\vec{a}\,)$ has the obvious meaning: it is the result of substituting a_i for x_i in θ, for each i in m. Then θ and $\mathfrak{A}$ determine the m-ary relation

$$\{\vec{a} \in A^m \colon \theta(\vec{a}\,)^{\mathfrak{A}_A} = 1\}$$

on A. Defining the interpretation of an open formula this way is like defining the interpretation of a term t as $\vec{a} \mapsto t(\vec{a}\,)^{\mathfrak{A}_A}$. As terms allow also another approach, shown in (‡), so with open formulas we can proceed recursively as follows. We have defined $\varphi^{\mathfrak{A}}$ when φ is atomic. Suppose we have defined $\varphi^{\mathfrak{A}}$ and $\psi^{\mathfrak{A}}$ for two m-ary open formulas φ and ψ. Then

$$\begin{aligned} \neg\varphi^{\mathfrak{A}} &= (\varphi^{\mathfrak{A}})^{\mathrm{c}} = A^m \smallsetminus \varphi^{\mathfrak{A}}; \\ (\varphi \mathbin{\&} \psi)^{\mathfrak{A}} &= \varphi^{\mathfrak{A}} \cap \psi^{\mathfrak{A}}. \end{aligned} \tag{†}$$

By induction,

$$\theta^{\mathfrak{A}} = \{\vec{a} \in A^m \colon \theta(\vec{a}\,)^{\mathfrak{A}_A} = 1\}$$

for all open θ.

Yet another way to understand interpretations of open formulas is by the equation

$$\mathbf{F}(\varphi_0, \dots, \varphi_{n-1})^{\mathfrak{A}} = \widehat{\mathbf{F}}({\varphi_0}^{\mathfrak{A}}, \dots, {\varphi_{n-1}}^{\mathfrak{A}}),$$

where the right-hand side has the same formal definition as $\widehat{\mathbf{F}}(\vec{e}\,)$ in §2.4, with adding 1 in $\mathbb{B}$ now replaced with complementation in $\mathcal{P}(A^m)$, and multiplication in $\mathbb{B}$ replaced with intersection in $\mathcal{P}(A^m)$.

3.4. Formulas in general

Formulas in general may contain the **existential quantifier** $\exists$. The inductive definition of **formulas** is:

(1) atomic formulas are formulas;
(2) if φ and ψ are formulas, then so are $\neg\varphi$ and $(\varphi \mathbin{\&} \psi)$;
(3) if φ is a formula, and x is a variable, then $\exists x\ \varphi$ is a formula.

The possibility of defining the foregoing interpretations of open formulas depends on the following.

Theorem 3.4.1 (Unique Readability). *Every formula of $\mathcal{L}$ is* uniquely *one of the following:*

(1) $t_0 = t_1$, for some terms t_e of $\mathcal{L}$;
(2) $Rt_0 \cdots t_{n-1}$ for some terms t_k and n-ary predicate R of $\mathcal{L}$, for some positive n;
(3) $\neg\varphi$ for some formula φ;
(4) $(\varphi \mathbin{\&} \psi)$ for some formulas φ and ψ;
(5) $\exists x\ \varphi$ for some formula φ and some variable x. □

In order to define interpretations of arbitrary formulas, we can still use (†) above to define $\neg\varphi^{\mathfrak{A}}$ and $(\varphi \mathbin{\&} \psi)^{\mathfrak{A}}$ in terms of $\varphi^{\mathfrak{A}}$ and $\psi^{\mathfrak{A}}$. We should also define $(\exists x\ \varphi)^{\mathfrak{A}}$ in terms of $\varphi^{\mathfrak{A}}$; but for this, we need a notion of *arity* of arbitrary formulas. Ultimately, if φ is $(n+1)$-ary, and x is x_n, then we shall have

$$\vec{a} \in (\exists x\ \varphi)^{\mathfrak{A}} \iff (\vec{a}, b) \in \varphi^{\mathfrak{A}} \text{ for some } b \text{ in } A. \qquad (\ddagger)$$

But complications arise if x is x_k, where $k < n$. When one takes care of these things, then, for every n-ary formula φ of $\mathcal{L}$, there will be an n-ary relation $\varphi^{\mathfrak{A}}$ on A; this relation is **defined by** φ, and the relation can be called a **0-definable** relation of $\mathfrak{A}$. The **definable** relations are those defined by formulas of $\mathcal{L}(A)$; more generally, if $X \subseteq A$, then the X-definable relations are those defined by formulas of $\mathcal{L}(X)$. (Singulary definable relations can just be called **definable sets.**)

If X and Y are k-ary definable relations of $\mathfrak{A}$, then so are X^{c}, $X \cap Y$, $X \cup Y$, &c. In short, all **Boolean combinations** of definable relations are definable, since we work in an adequate signature for propositional logic.

If φ is an n-ary formula, defining as such the n-ary relation X, then we can also treat φ as $(n+1)$-ary, defining the relation $X \times A$ on A. This relation is the set

$$\{(\vec{x}, y) \in A^{n+1} \colon \vec{x} \in X\}.$$

This set is also $\pi^{-1}[X]$, where π is the function

$$(\vec{x}, y) \longmapsto \vec{x} \qquad (\S)$$

from A^{n+1} to A^n; this function is **projection** onto the first n coordinates. In short then, *inverse images* of definable sets under projections are definable. By (‡), *images* under projections will be definable.

Let the set of formulas of $\mathcal{L}$ be

$$\mathrm{Fm}_{\mathcal{L}}.$$

We recursively define a function

$$\varphi \longmapsto \mathrm{fv}(\varphi)$$

from $\mathrm{Fm}_{\mathcal{L}}$ to $\mathcal{P}(\{\mathsf{x}_k \colon k \in \omega\})$ as follows:

(1) $\mathrm{fv}(\alpha)$ is the set of variables in α, if α is atomic;
(2) $\mathrm{fv}((\varphi \mathbin{\&} \psi)) = \mathrm{fv}(\varphi) \cup \mathrm{fv}(\psi)$;
(3) $\mathrm{fv}(\exists x\ \varphi) = \mathrm{fv}(\varphi) \smallsetminus \{x\}$.

Then $\mathrm{fv}(\varphi)$ is the set of **free variables** of φ. If $\mathrm{fv}(\varphi) = \varnothing$, then φ is a **sentence;** the set of sentences of $\mathcal{L}$ can be denoted by

$$\mathrm{Sn}_{\mathcal{L}}\,.$$

So an atomic sentence α is a nullary atomic formula; in this case, we can define

$$\mathfrak{A} \models \alpha \iff \alpha^{\mathfrak{A}} = 1; \tag{¶}$$

in either case, α is **true** in $\mathfrak{A}$. Otherwise, α is **false in** $\mathfrak{A}$, and we can write

$$\mathfrak{A} \nvDash \alpha.$$

We can also define

$$\begin{aligned} \mathfrak{A} \models \neg\sigma &\iff \mathfrak{A} \nvDash \sigma;\\ \mathfrak{A} \models \sigma \mathbin{\&} \tau &\iff \mathfrak{A} \models \sigma \ \ \&\ \ \mathfrak{A} \models \tau; \end{aligned} \tag{‖}$$

provided σ and τ are sentences for which truth and falsity in $\mathfrak{A}$ have been defined. To define $\mathfrak{A} \models \exists x\ \varphi$, we need a notion of *substitution,* whereby to convert φ to a sentence; but then we should assume that we have been working with formulas of $\mathcal{L}(A)$ all along. For formulas φ, if x is a variable and t is a term, we define the formula

$$(\varphi)^x_t$$

recursively:

(1) If α is atomic, then $(\alpha)^x_t$ is the result of substituting t for x in α;
(2) $(\neg\varphi)^x_t$ is $\neg(\varphi)^x_t$;
(3) $((\varphi \mathbin{\&} \psi))^x_t$ is $((\varphi)^x_t \mathbin{\&} (\psi)^x_t)$;
(4) $(\exists x\ \varphi)^x_t$ is $\exists x\ \varphi$ (no change);
(5) $(\exists u\ \varphi)^x_t$ is $\exists u\ (\varphi)^x_t$, if u is not x.

Then $(\varphi)^x_t$ is the result of replacing each **free occurrence** of x in φ with t. Now we can define

$$\mathfrak{A} \models \exists x\ \varphi \iff \mathfrak{A} \models \varphi^x_a \text{ for some } a \text{ in } A. \tag{$**$}$$

We have now completed the definition of truth; it is expressed by $(*)$, (¶), (‖), and $(**)$.

If $\mathrm{fv}(\varphi) \subseteq \{\mathsf{x}_k \colon k < n\}$, then φ can be called n-ary, and we can write φ as

$$\varphi(\mathsf{x}_0, \dots, \mathsf{x}_{n-1}).$$

Then, instead of $(\cdots((\varphi)^{\mathsf{x}_0}_{a_0})\cdots)^{\mathsf{x}_{n-1}}_{a_{n-1}}$, we can write

$$\varphi(a_0, \dots, a_{n-1})$$

or $\varphi(\vec{a}\,)$. Note well that $\vec{a}$ is a tuple of *constants.* We can let it be a tuple $(t_0, \dots, t_{n-1})$ of arbitrary terms; but then we must ensure that $\varphi(t_0, \dots, t_{n-1})$ is the result of *simultaneously* substituting each t_k for the free instances of the corresponding variable x_k. Now we can define

$$\varphi^{\mathfrak{A}} = \{\vec{a} \in A^n \colon \mathfrak{A} \models \varphi(\vec{a}\,)\}$$

for all formulas φ.

Theorem 3.4.2. *Let φ be an n-ary formula of $\mathcal{L}$.*

(1) *If φ is $\neg\psi$, then $\varphi^{\mathfrak{A}} = A^n \smallsetminus \psi^{\mathfrak{A}}$.*
(2) *If φ is $(\chi \mathbin{\&} \psi)$, then $\varphi^{\mathfrak{A}} = \chi^{\mathfrak{A}} \cap \psi^{\mathfrak{A}}$.*
(3) *If φ is $\exists \mathsf{x}_n\ \psi$, then $\varphi^{\mathfrak{A}} = \pi[\psi^{\mathfrak{A}}]$, where π is as in (§).* □

If $\mathrm{fv}(\varphi) = \{u_0, \dots, u_{n-1}\}$, and

$$\mathfrak{A} \vDash \exists u_0\ \cdots \exists u_{n-1}\ \varphi,$$

then φ is **satisfied** in $\mathfrak{A}$, or $\mathfrak{A}$ **satisfies** φ.

In a formula of $\mathcal{L}(A)$, any constants from A can be called **parameters.** So the definable relations of $\mathfrak{A}$ are, more fully, the relations definable *with parameters. Algebraic geometry* studies the definable relations of $\mathbb{C}$ and of other fields. See Appendix G for more on definable sets in general.

Strictly, interpretations and the truth-relation $\vDash$ have been defined for a particular signature. However, expanding a structure does not change the interpretation of a formula or the truth-value of a sentence in that structure.

Theorem 3.4.3. *Say $\mathcal{L} \subseteq \mathcal{L}'$, and $\mathfrak{A}$ is an $\mathcal{L}$-structure with an expansion $\mathfrak{A}'$ to $\mathcal{L}'$. If φ is a formula of $\mathcal{L}$, then*

$$\varphi^{\mathfrak{A}'} = \varphi^{\mathfrak{A}}.$$

In particular, if σ is a sentence of $\mathcal{L}$, then

$$\mathfrak{A}' \vDash \sigma \iff \mathfrak{A} \vDash \sigma.$$

Proof. The definition of the interpretation of φ involves no symbols in $\mathcal{L}' \smallsetminus \mathcal{L}$ or their interpretations, so $\vec{a} \in \varphi^{\mathfrak{A}'} \iff \vec{a} \in \varphi^{\mathfrak{A}}$. □

Two sentences are **(logically) equivalent** if each is a logical consequence of the other. We define subformulas as in propositional logic, and we have analogues of Lemmas 2.4.1 and 2.4.2, and hence

Theorem 3.4.4 (Replacement). *If φ is a subformula of ψ, and φ' is another formula, then the result of replacing φ with φ' in ψ is a formula ψ'. If also φ is equivalent to φ', then ψ is equivalent to ψ'.* □

We can use $(\varphi \Rightarrow \psi)$ as equivalent to $\neg(\varphi \mathbin{\&} \neg\psi)$, and $(\varphi \Leftrightarrow \psi)$ as equivalent to $((\varphi \Rightarrow \psi) \mathbin{\&} (\psi \Rightarrow \varphi))$.

Theorem 3.4.5.

(1) *Two sentences σ and τ of $\mathcal{L}$ are equivalent if and only if $\vDash (\sigma \Leftrightarrow \tau)$.*
(2) *Logical equivalence is an equivalence-relation on $\mathrm{Sn}_{\mathcal{L}}$.* □

3.5. Logical entailment

Having defined the truth-relation $\vDash$ for first-order logic, we have related notions, as in propositional logic. If Σ is a set of sentences, each of which is true in $\mathfrak{A}$, then $\mathfrak{A}$ is a **model** of Σ, and we write

$$\mathfrak{A} \vDash \Sigma.$$

The $\mathcal{L}$-structures that are models of Σ compose

$$\mathrm{Mod}_{\mathcal{L}}(\Sigma);$$

if this is nonempty, then it is a *proper class* (see §1.6). The **theory** of a class $\mathbf{K}$ of structures is the set

$$\mathrm{Th}(\mathbf{K})$$

of sentences that are true in each structure in $\mathbf{K}$. If $\mathbf{K}$ has a single element, $\mathfrak{A}$, then $\mathrm{Th}(\mathbf{K})$ can be written as

$$\mathrm{Th}(\mathfrak{A}).$$

Hence

$$\sigma \in \mathrm{Th}(\mathfrak{A}) \iff \mathfrak{A} \vDash \sigma.$$

The set of **logical consequences** of a set Σ of sentences is $\mathrm{Th}(\mathrm{Mod}_{\mathcal{L}}(\Sigma))$, also denoted by

$$\mathrm{Con}_{\mathcal{L}}(\Sigma).$$

Then $\mathrm{Con}_{\mathcal{L}}(\Sigma)$ is always a theory, namely the theory of $\mathrm{Mod}_{\mathcal{L}}(\Sigma)$. We now have analogues of Lemma 2.5.2 and Theorem 2.5.1.

If $\sigma \in \mathrm{Con}_{\mathcal{L}}(\Sigma)$, then Σ **logically entails** σ. The logical consequences of $\varnothing$ are the **validities.** In particular, if $\mathbf{F}$ is a tautology of propositional logic, and σ_i are sentences of $\mathcal{L}$, then $\mathbf{F}(\tau_0, \dots, \tau_{n-1})$ is a validity of $\mathcal{L}$. Such a validity is also called a **tautology;** but there are validities that are not tautologies, for example $\neg\exists x\ x \neq x$.

Instead of $\neg\exists v\ \varphi$, we may write

$$\forall v\ \neg\varphi.$$

Here $\forall$ is the **universal quantifier.** Then $\neg\forall v\ \varphi$ is equivalent to $\exists v\ \neg\varphi$. Let P and Q be singulary predicates. To prove that the sentence

$$(\forall x\ (Px \Rightarrow Qx) \Rightarrow (\forall x\ Px \Rightarrow \forall x\ Qx)) \tag{$*$}$$

is a validity, it is enough to show that $\mathfrak{A} \vDash (\forall x\ Px \Rightarrow \forall x\ Qx)$ whenever $\mathfrak{A} \vDash \forall x\ (Px \Rightarrow Qx)$. So suppose

$$\mathfrak{A} \vDash \forall x\ (Px \Rightarrow Qx). \tag{$\dagger$}$$

It is now enough to show that, if also $\mathfrak{A} \vDash \forall x\ Px$, then $\mathfrak{A} \vDash \forall x\ Qx$. So suppose

$$\mathfrak{A} \vDash \forall x\ Px. \tag{$\ddagger$}$$

Let $a \in A$. Then $\mathfrak{A} \vDash Pa$, by ($\ddagger$). But $\mathfrak{A} \vDash (Pa \Rightarrow Qa)$, by ($\dagger$). Hence $\mathfrak{A} \vDash Qa$. Since a was arbitrary, we have $\mathfrak{A} \vDash \forall x\ Qx$. Therefore ($*$) is a validity.

The theory $\mathrm{Con}_{\mathcal{L}}(\Sigma)$ is **axiomatizeed** by Σ; the elements of Σ are **axioms** for this theory. If $\mathrm{fv}(\varphi) = \{u_0, \dots, u_{n-1}\}$, then the sentence $\forall u_0 \cdots \forall u_{n-1}\ \varphi$ is a **generalization** of φ. We may use formulas to denote their generalizations. For example, group-theory in the signature $\{1, {}^{-1}, \cdot\}$ is axiomatized by the (generalizations of the) following formulas.

$$\begin{gathered} x \cdot (y \cdot z) = (x \cdot y) \cdot z, \\ x \cdot 1 = x, \qquad x \cdot x^{-1} = 1, \\ 1 \cdot x = x, \qquad x^{-1} \cdot x = 1. \end{gathered} \tag{§}$$

If Σ has no models, then $\mathrm{Con}_{\mathcal{L}}(\Sigma) = \mathrm{Sn}_{\mathcal{L}}$. Thus $\mathrm{Sn}_{\mathcal{L}}$ is the only theory (of $\mathcal{L}$) with no models. A theory with models is **complete** if, for every sentence σ of its signature, the theory contains either σ or $\neg\sigma$ (hence exactly one of these).

THEOREM 3.5.1. *Every theory* $\mathrm{Th}(\mathfrak{A})$ *is complete. Every complete theory is* $\mathrm{Th}(\mathfrak{A})$ *for some* $\mathfrak{A}$.

PROOF. Since

$$\sigma \in \mathrm{Th}(\mathfrak{A}) \iff \mathfrak{A} \models \sigma \iff \mathfrak{A} \not\models \neg\sigma \iff \neg\sigma \notin \mathrm{Th}(\mathfrak{A}),$$

$\mathrm{Th}(\mathfrak{A})$ is complete. If T is a complete theory, then in particular it has a model $\mathfrak{A}$. Then $T \subseteq \mathrm{Th}(\mathfrak{A})$; but also

$$\sigma \notin T \implies \neg\sigma \in T \implies \mathfrak{A} \models \neg\sigma \implies \neg\sigma \in \mathrm{Th}(\mathfrak{A}) \implies \sigma \notin \mathrm{Th}(\mathfrak{A}),$$

by completeness of T, and hence $\mathrm{Th}(\mathfrak{A}) \subseteq T$. □

Exercises

EXERCISE 3.1.

(1) Prove the Associativity Theorem, 3.1.1.
(2) What does this theorem have to do with Theorems C.1(2) and C.3(3)?

EXERCISE 3.2. Prove Theorem 3.4.1.

EXERCISE 3.3. Prove Theorem 3.4.2.

EXERCISE 3.4. Let $\mathcal{L} = \{R\}$, where R is a binary predicate, and let $\mathfrak{A}$ be the $\mathcal{L}$-structure $(\mathbb{Z}, \leqslant)$. Determine $\varphi^{\mathfrak{A}}$ if φ is:

(1) $\forall x_1\ (Rx_1x_0 \Rightarrow Rx_0x_1)$;
(2) $\forall x_2\ (Rx_2x_0 \lor Rx_1x_2)$.

EXERCISE 3.5. Let $\mathcal{L}$ be $\{S, P\}$, where S and P are binary function-symbols. Then $(\mathbb{R}, +, \cdot)$ is an $\mathcal{L}$-structure. Show that the following sets and relations are definable in this structure:

(1) $\{0\}$;
(2) $\{1\}$;
(3) $\{x \in \mathbb{R} \colon 0 < x\}$;
(4) $\{(x, y) \in \mathbb{R}^2 \colon x < y\}$.

EXERCISE 3.6. Show that the following sets are definable in $(\omega, +, \cdot, \leqslant, 0, 1)$:

(1) the set of even numbers;
(2) the set of prime numbers.

EXERCISE 3.7. Let R be the binary relation

$$\{(x, x+1) \colon x \in \mathbb{Z}\}$$

on $\mathbb{Z}$. Show that R is 0-definable in the structure $(\mathbb{Z}, <)$; that is, find a binary formula φ in the signature $\{<\}$ such that $\varphi^{(\mathbb{Z},<)} = R$.

EXERCISE 3.8. Find an *open* sentence that is a validity, but not a tautology.

EXERCISE 3.9. Prove the **Lemma on Constants:** Suppose Σ is a set of sentences of $\mathcal{L}$, and c_k are constants not in $\mathcal{L}$, and φ is an n-ary formula of $\mathcal{L}$. Then

$$\Sigma \models \forall \mathsf{x}_0 \ \cdots \forall \mathsf{x}_{n-1}\ \varphi \iff \Sigma \models \varphi(c_0, \ldots, c_{n-1}).$$

EXERCISE 3.10. Prove the Replacement Theorem, 3.4.4.

EXERCISE 3.11. Prove Theorem 3.4.5.

Exercise 3.12. Letting P and Q be singulary predicates, determine, from the definition of $\vDash$, whether the following hold.

(1) $\exists x\ Px \Rightarrow \exists x\ Qx \vDash \forall x\ (Px \Rightarrow Qx)$;
(2) $\forall x\ Px \Rightarrow \exists x\ Qx \vDash \exists x\ (Px \Rightarrow Qx)$;
(3) $\exists x\ (Px \Rightarrow Qx) \vDash \forall x\ Px \Rightarrow \exists x\ Qx$;
(4) $\{\exists x\ Px,\ \exists x\ Qx\} \vDash \exists x\ (Px\ \&\ Qx)$;
(5) $\exists x\ Px \Rightarrow \exists y\ Qy \vDash \forall x\ \exists y\ (Px \Rightarrow Qy)$.

Exercise 3.13. Axiomatize group-theory in the signature $\{\,\cdot\,\}$.

CHAPTER 4

Quantifier-elimination and complete theories

It is easy to show that a theory is *not* complete. For example, the theory of groups is not complete, since the sentence

$$\forall x\ \forall y\ x \cdot y = y \cdot x$$

is true only in abelian groups (by definition), but there are non-abelian groups (such as the group of permutations of three objects). The theory of abelian groups is not complete either, since (in the signature $\{+, -, 0\}$) the sentence

$$\forall x\ (x + x = 0 \Rightarrow x = 0)$$

is true in $(\mathbb{Z}, +, -, 0)$, but false in $(\mathbb{Z}/2\mathbb{Z}, +, -, 0)$.

To show that a theory *is* complete, there are various methods that can be tried. One of these is *elimination of quantifiers,* which we shall perform in two examples.

4.1. Total orders

Let TO be the theory of strict total orders; this is axiomatized by the generalizations of:

$$x \not< x, \qquad x < y \ \&\ y < z \Rightarrow x < z,$$
$$x < y \Rightarrow y \not< x, \qquad x < y \lor y < x \lor x = y.$$

This theory is not complete, since $(\omega, <)$ and $(\mathbb{Z}, <)$ are models of TO with different complete theories (exercise).

Let TO^* be the theory of **dense total orders without endpoints:** this means TO^* has the axioms of TO, along with the generalizations of:

$$\exists z\ (x < z \ \&\ z < y), \qquad \exists y\ y < x, \qquad \exists y\ x < y.$$

The theory TO^* has a model, namely $(\mathbb{Q}, <)$. We shall show that TO^* is complete, hence equal to $\mathrm{Th}(\mathbb{Q}, <)$.

Two formulas φ and ψ are **equivalent *modulo*** a theory T, or equivalent **in T,** if

$$T \vDash \varphi \Leftrightarrow \psi.$$

Then T admits **(full) elimination of quantifiers** if, for every formula, there is an *open* formula that is equivalent to it in T.

LEMMA 4.1.1. *An $\mathcal{L}$-theory T admits quantifier-elimination, provided that, if φ is an open formula, and v is a variable, then $\exists v\ \varphi$ is equivalent* modulo T *to an open formula.*

PROOF. Use induction on formulas. Every atomic formula is equivalent in T to an open formula, namely itself. Now suppose φ and ψ are equivalent in T to open formulas α and β respectively. Then

$$T \vDash \neg\varphi \Leftrightarrow \neg\alpha, \qquad T \vDash \varphi \ \&\ \psi \Leftrightarrow \alpha \ \&\ \beta;$$

but $\neg\alpha$ and $\alpha \Rightarrow \beta$ are open. Finally, $T \vDash \exists v\ \varphi \Leftrightarrow \exists v\ \alpha$ (exercise); but by assumption, $\exists v\ \alpha$ is equivalent to an open formula γ; so $T \vDash \exists v\ \varphi \Leftrightarrow \gamma$ (exercise). This completes the induction. □

The lemma can be improved slightly. First, if Σ is a set $\{\varphi_k \colon k < n\}$ of formulas, then for the **disjunction** $\varphi_0 \vee \varphi_1 \vee \cdots \vee \varphi_{n-1}$, we may write simply

$$\bigvee_{k<n} \varphi_k,$$

or even $\bigvee \Sigma$. The order in which the φ_k appear in the original disjunction is unimportant. Likewise, the **conjunction** $\varphi_0 \mathbin{\&} \varphi_1 \mathbin{\&} \cdots \mathbin{\&} \varphi_{n-1}$ can be denoted by

$$\bigwedge_{k<n} \varphi_k,$$

or even $\bigwedge \Sigma$.

Theorem 4.1.1. *Every open formula is logically equivalent to a formula*

$$\bigvee_{i<m} \bigwedge \Sigma_i \tag{$*$}$$

where Σ_i is $\{\alpha_i^{(j)} \colon j < n\}$, and each $\alpha_i^{(j)}$ is an atomic or a negated atomic formula.

Proof. Exercise. □

The formula in $(*)$ is in **disjunctive normal form.** Note that

$$\vDash \exists v \bigvee_{i<m} \bigwedge \Sigma_i \Leftrightarrow \bigvee_{i<m} \exists v\ \bigwedge \Sigma_i \tag{$\dagger$}$$

(**exercise**). The formulas $\exists v\ \bigwedge \Sigma_i$ are said to be *primitive.* In general, a **primitive formula** is a formula

$$\exists u_0\ \cdots \exists u_{n-1}\ \bigwedge \Sigma,$$

where Σ is a finite non-empty set of atomic and negated atomic formulas. Using $(\dagger)$, we can adjust the induction in the proof above to show

Lemma 4.1.2. *A theory admits quantifier-elimination, provided that every primitive formula with one (existential) quantifier is equivalent,* modulo *the theory, to an open formula.* □

Henceforth suppose $\mathcal{L}$ is $\{<\}$, and $\mathrm{TO} \subseteq T$; so T is *a* theory of total orders. Then we can improve Lemma 4.1.1 even more. Indeed, the atomic formulas of $\mathcal{L}$ now are $x = y$ and $x < y$, where x and y are variables. Moreover,

$$\mathrm{TO} \vDash x \not< y \Leftrightarrow x = y \vee y < x,$$
$$\mathrm{TO} \vDash x \neq y \Leftrightarrow x < y \vee y < x.$$

Hence, in $\mathcal{L}$, any formula is equivalent, *modulo* TO, to the result of replacing each negated atomic subformula with the appropriate disjunction of atomic formulas. If this replacement is done to a formula in disjunctive normal form, then the new formula will have a disjunctive normal form that involves no negations. So T admits quantifier-elimination, provided that every formula

$$\exists v\ \bigwedge \Sigma$$

is equivalent, *modulo* T, to an open formula, where now Σ is a set of atomic formulas. Using this criterion, we show:

THEOREM 4.1.2. TO* *admits elimination of quantifiers.*

PROOF. Let Σ be a finite, non-empty set of atomic formulas of $\mathcal{L}$. We shall eliminate the quantifier from the formula $\exists v\ \bigwedge \Sigma$. Let X be a set $\{\mathsf{x}_0, \dots, \mathsf{x}_n\}$ containing all variables appearing in formulas in Σ. Suppose $\mathfrak{A}$ is an $\mathcal{L}$-structure, and $\vec{a} \in A^{n+1}$. Then we can let

$$\Sigma(\vec{a}\,) = \{\alpha(\vec{a}\,) \colon \alpha \in \Sigma\}.$$

Suppose in fact

$$\mathfrak{A} \vDash \mathrm{TO} \cup \{\bigwedge \Sigma(\vec{a}\,)\}.$$

Let us define $\Sigma_{(\mathfrak{A},\vec{a}\,)}$ as the set of atomic formulas α such that $\mathrm{fv}(\alpha) \subseteq X$ and $\mathfrak{A} \vDash \alpha(\vec{a}\,)$. Then

$$\Sigma \subseteq \Sigma_{(\mathfrak{A},\vec{a}\,)}.$$

Moreover, once Σ has been chosen, *there are only finitely many possibilities for the set* $\Sigma_{(\mathfrak{A},\vec{a}\,)}$. Let us list these possibilities as

$$\Sigma_0, \dots, \Sigma_{m-1}.$$

Now, possibly $m = 0$ here. In this case,

$$\mathrm{TO} \vDash \exists v\ \bigwedge \Sigma \Leftrightarrow v \neq v,$$

so we are done. Henceforth we may assume $m > 0$. If $\mathfrak{B} \vDash \mathrm{TO} \cup \{\bigwedge \Sigma(\vec{b}\,)\}$, then

$$\mathfrak{B} \vDash \bigwedge \Sigma_i(\vec{b}\,)$$

for some i in m. Therefore

$$\mathrm{TO} \vDash \bigwedge \Sigma \Leftrightarrow \bigvee_{i<m} \bigwedge \Sigma_i,$$

and hence

$$\mathrm{TO} \vDash \exists v\ \bigwedge \Sigma \Leftrightarrow \bigvee_{i<m} \exists v\ \bigwedge \Sigma_i.$$

Therefore, for our proof of quantifier-elimination, we may assume that Σ *is* one of the sets $\Sigma_{(\mathfrak{A},\vec{a}\,)}$ (so that, in particular, $m = 1$).

Now partition Σ as $\Gamma \cup \Delta$, where no formula in Γ, but every formula in Δ, contains v. There are two extreme possibilities, where one of these sets is empty. Suppose first $\Gamma = \varnothing$. Then $X = \{v\}$ (since if $x \in X \smallsetminus \{v\}$, then Γ contains $x = x$). Also, $\Sigma = \Delta = \{v = v\}$, so

$$\vDash \exists v\ \bigwedge \Sigma \Leftrightarrow v = v,$$

and we are done in this case. Now suppose $\Delta = \varnothing$. Then $v \notin X$, and

$$\vDash \exists v\ \bigwedge \Sigma \Leftrightarrow \bigwedge \Sigma,$$

so we are done in *this* case. Henceforth, suppose neither Γ nor Δ is empty. Then

$$\vDash \exists v\ \bigwedge \Sigma \Leftrightarrow \bigwedge \Gamma \mathbin{\&} \exists v\ \bigwedge \Delta.$$

We shall show that

$$\mathrm{TO}^* \vDash \exists v\ \bigwedge \Sigma \Leftrightarrow \bigwedge \Gamma, \tag{‡}$$

which will complete the proof. To show (‡), it is enough to show

$$\mathrm{TO}^* \vDash \bigwedge \Gamma \Rightarrow \exists v \ \bigwedge \Delta.$$

Since Σ is $\Sigma_{(\mathfrak{A},\vec{a})}$, we have for all i and j in $n+1$ that $a_i < a_j$ if and only if Σ contains $\mathsf{x}_i < \mathsf{x}_j$, and $a_i = a_j$ if and only if Σ contains $x_i = x_j$. We also have $v \in X$. We can relabel the elements of X as necessary so that v is x_n and

$$a_0 \leqslant \cdots \leqslant a_{n-1}.$$

Suppose $\mathfrak{B} \vDash \mathrm{TO}^*$, and B^n contains $\vec{b}$ such that $\mathfrak{B} \vDash \bigwedge \Gamma(\vec{b})$. We have to show that there is c in B such that $\mathfrak{B} \vDash \bigwedge \Delta(\vec{b}, c)$. Now, for all i and j in n, we have

$$b_i < b_j \iff a_i < a_j, \qquad\qquad b_i = b_j \iff a_i = a_j.$$

Because $\mathfrak{B}$ is a model of TO^* (and not just TO), we can find c as needed according to the relation of a_n with the other a_i:

(1) If $a_n = a_i$ for some i in n, then let $c = b_i$.
(2) If $a_{n-1} < a_n$, then let c be greater than b_{n-1}.
(3) If $a_n < a_0$, then let c be less than b_0.
(4) If $a_k < a_n < a_{k+1}$, then we can let c be such that $b_k < c < b_{k+1}$.

This completes the proof that TO^* admits quantifier-elimination. □

In the proof, we can let X be *precisely* the set of variables appearing in Σ. Then we have that, *modulo* TO^*, the formula $\exists v \ \bigwedge \Sigma$ is equivalent to $v \neq v$ or $v = v$ or an open formula *with the same free variables as* $\exists v \ \bigwedge \Sigma$. In the signature $\{<\}$, there are no open sentences. Therefore, *modulo* TO^*, every sentence is equivalent to $v \neq v$ or $v = v$. The former is an **absurdity** (the negation of a validity), which we can denote by $\bot$; and $v = v$ is a validity, which we can denote by $\top$.

THEOREM 4.1.3. TO^* *is a complete theory.*

PROOF. As we have just noted, every sentence is equivalent to an absurdity or a validity. Suppose $\mathrm{TO}^* \vDash \sigma \Leftrightarrow \bot$. But $\vDash (\sigma \Leftrightarrow \bot) \Leftrightarrow \neg\sigma$; so $\mathrm{TO}^* \vDash \neg\sigma$. Similarly, if $\mathrm{TO}^* \vDash \sigma \Leftrightarrow \top$, then $\mathrm{TO}^* \vDash \sigma$. Hence, for all sentences σ, if $\mathrm{TO}^* \nvDash \sigma$, then $\mathrm{TO}^* \vDash \neg\sigma$. Therefore TO^* is complete by Theorem 3.5.1. □

4.2. The natural numbers

Let us now understand the signature of iterative structures (in the sense of §1.4) as $\{0, {}'\}$. In this signature, let Itr be the theory axiomatized by (the generalizations of)

$$\begin{aligned} x' &\neq 0, \\ x' = y' &\Rightarrow x = y; \end{aligned} \tag{$*$}$$

let Itr^* be the theory with the same two axioms and one more, namely

$$\exists y \ (x = 0 \lor y' = x). \tag{†}$$

Note that all models of Itr that admit induction are isomorphic to ω; models of Itr^* satisfy one *consequence* of induction, namely Theorem 1.4.1.

An **embedding** of algebras is an injective homomorphism.

LEMMA 4.2.1. *For every model* $\mathfrak{A}$ *of* Itr, *for every* b *in* A, *the unique homomorphism from* ω *into* $(A, b, {}')$ *is an embedding.* □

THEOREM 4.2.1. *The theory* Itr* *admits elimination of quantifiers.*

PROOF. Instead of $x''^{\cdots}{}'$ with n primes, let us write $x^{(n)}$. Let Σ be a non-empty finite set of atomic and negated atomic formulas. If the variable v does not appear in any formula in Σ, then $\exists v\ \bigwedge\Sigma$ is equivalent *modulo* Itr* to $\bigwedge\Sigma$. So suppose it does appear. We consider the case where v is the *only* variable in some formula in Σ. If

$$\Sigma = \Lambda \cup \{v^{(m)} = v^{(m)}\},$$

then $\exists v\ \bigwedge\Sigma$ is equivalent in Itr to $\exists v\ \bigwedge\Lambda$, while if Σ contains $v^{(m)} \neq v^{(m)}$, then $\exists v\ \bigwedge\Sigma$ is equivalent to $v \neq v$. By Lemma 4.2.1, if Σ contains $v^{(m)} = v^{(n)}$ where $m \neq n$, then $\exists v\ \bigwedge\Sigma$ is equivalent to $v \neq v$, while if $\Sigma = \Lambda \cup \{v^{(m)} \neq v^{(n)}\}$ where $m \neq n$, then $\exists v\ \bigwedge\Sigma$ is equivalent to $\exists v\ \bigwedge\Lambda$.

We may now assume that v never appears on both sides of an equation or inequation in Σ. But suppose $\Sigma = \Lambda \cup \{v^{(m)} = t^{(n)}\}$, where t is a term not featuring v. By the injectivity of $x \mapsto x'$ ensured by $(*)$, we may assume that m or n is 0. Suppose first $m = 0$. Let Λ^* be the result of replacing each v appearing in a formula in Λ with $t^{(n)}$. Then

$$\begin{aligned}\mathrm{Itr}^* \vDash \exists v\ \bigwedge\Sigma &\iff \exists v\ \Big(\bigwedge\Lambda^* \ \&\ v = t^{(n)}\Big)\\ &\iff \bigwedge\Lambda^* \ \&\ \exists v\ v = t^{(n)}\\ &\iff \bigwedge\Lambda^*.\end{aligned}$$

Now suppose instead $n = 0$. Whenever v appears in an equation or inequation in Λ, we can add primes to both sides until v has at least m primes on it: the resulting formula is equivalent to the original formula. Then we can replace $v^{(m)}$ with t. Thus we get Λ^* such that

$$\begin{aligned}\mathrm{Itr}^* \vDash \exists v\ \bigwedge\Sigma &\iff \exists v\ \Big(\bigwedge\Lambda^* \ \&\ v^{(m)} = t\Big)\\ &\iff \bigwedge\Lambda^* \ \&\ \exists v\ v^{(m)} = t.\end{aligned}$$

But by (†) we have also

$$\mathrm{Itr}^* \vDash \exists v\ v^{(m)} = t \Leftrightarrow \bigwedge_{k<m} t \neq 0^{(k)}.$$

In the final case, v appears only in *inequations* in Σ (and only on one side of each of these), not in equations. Let Σ^* be the result of deleting those inequations. Then $\exists v\ \bigwedge\Sigma$ is equivalent to $\bigwedge\Sigma^*$. To see this, note first that all models of Itr* are infinite. We may assume that the variables in Σ form the list $(\mathsf{x}_0, \dots, \mathsf{x}_n)$, where x_n is v, so that the variables in Σ^* are on the list $(\mathsf{x}_0, \dots, \mathsf{x}_{n-1})$. Let $\mathfrak{A} \vDash \mathrm{Itr}^*$ and $\mathfrak{A} \vDash \Sigma^*(\vec{a}\,)$ for some $\vec{a}$ from A. The inequations in $\Sigma(\vec{a}\,, v)$ involving v give a *finite* set of elements of A that v cannot be. Chose b from outside this set. Then $\mathrm{Itr}^* \vDash \Sigma(\vec{a}\,, b)$, so $\mathrm{Itr}^* \vDash \exists v\ \bigwedge\Sigma(\vec{a}\,, v)$. □

THEOREM 4.2.2. Itr* $= \mathrm{Th}(\omega, 0, ')$*; in particular,* Itr* *is complete.* □

Exercises

EXERCISE 4.1. Show that $(\omega, <)$ and $(\mathbb{Z}, <)$ have different complete theories.

EXERCISE 4.2. Supply the details of the proof of Lemma 4.1.1.

Exercise 4.3. Prove Lemma 4.2.1.

Exercise 4.4. Prove Theorem 4.2.2.

Exercise 4.5. Describe all models of Itr^*. In particular, find a one-to-one correspondence between the cardinal numbers and the isomorphism-classes of models of Itr^*. Why does this, together with Theorem 4.2.2, not contradict Theorem 1.4.3?

Exercise 4.6. Let A be an infinite set, that is, an infinite structure in the empty signature. Find a complete axiomatization of $\mathrm{Th}(A)$.

Exercise 4.7. Find a complete axiomatization of $\mathrm{Th}(\omega, 0, ', <)$, and describe all models of this theory.

Exercise 4.8. Describe all definable sets of a model of TO^*.

Exercise 4.9. Describe all definable sets of

(1) $(\omega, 0, ')$;
(2) an arbitrary model of Itr^*;
(3) $(\omega, 0, ', <)$;
(4) an arbitrary model of $\mathrm{Th}(\omega, 0, ', <)$.

CHAPTER 5

Relations between structures

Given a signature $\mathcal{L}$, we consider several relations on the class of $\mathcal{L}$-structures, namely

$$\mathrm{Mod}_{\mathcal{L}}(\varnothing).$$

Throughout this chapter, let $\mathfrak{A}$ and $\mathfrak{B}$ be arbitrary $\mathcal{L}$-structures.

5.1. Basic relations

If $h\colon A \to B$, then we may understand h also as the function from A^n to B^n given by

$$h(a_0, \dots, a_{n-1}) = (h(a_0), \dots, h(a_{n-1})). \tag{$*$}$$

Then $\mathfrak{A}$ and $\mathfrak{B}$ are **isomorphic,** and we write

$$\mathfrak{A} \cong \mathfrak{B},$$

if there is a bijection h from A to B such that

(1) $h(c^{\mathfrak{A}}) = c^{\mathfrak{B}}$ for all constants c in $\mathcal{L}$,
(2) $h \circ f^{\mathfrak{A}} = f^{\mathfrak{B}} \circ h$ for all function-symbols f in $\mathcal{L}$,
(3) $h[R^{\mathfrak{A}}] = R^{\mathfrak{B}}$ for all predicates in $\mathcal{L}$.

In this case, we may write

$$h\colon \mathfrak{A} \xrightarrow{\cong} \mathfrak{B},$$

calling h an **isomorphism.** This is also the name of the relation $\cong$, which is an equivalence-relation on $\mathrm{Mod}_{\mathcal{L}}(\varnothing)$.

Another equivalence-relation on $\mathrm{Mod}_{\mathcal{L}}(\varnothing)$ is **elementary equivalence.** The two structures $\mathfrak{A}$ and $\mathfrak{B}$ are **elementarily equivalent,** and we write

$$\mathfrak{A} \equiv \mathfrak{B},$$

if they have the same theory, that is, $\mathrm{Th}(\mathfrak{A}) = \mathrm{Th}(\mathfrak{B})$.

The notion of *substructure* was introduced in §1.7. We say that $\mathfrak{A}$ is a **substructure** of $\mathfrak{B}$, and $\mathfrak{B}$ is an **extension** of $\mathfrak{A}$, and we write

$$\mathfrak{A} \subseteq \mathfrak{B},$$

if $A \subseteq B$ and

(1) $c^{\mathfrak{A}} = c^{\mathfrak{B}}$ for all constants c of $\mathcal{L}$,
(2) $f^{\mathfrak{A}} = f^{\mathfrak{B}} \circ \mathrm{id}_{A^n}$ for all n-ary function-symbols f of $\mathcal{L}$, for all positive n in ω,
(3) $R^{\mathfrak{A}} = A^n \cap R^{\mathfrak{B}}$ for all n-ary predicates R of $\mathcal{L}$, for all positive n in ω.

Then $\subseteq$ is a reflexive ordering of $\mathrm{Mod}_{\mathcal{L}}(\varnothing)$.

5.2. Derived relations

If $\mathfrak{A} \subseteq \mathfrak{B}$ and $\mathfrak{A}_A \equiv \mathfrak{B}_A$, then $\mathfrak{A}$ is an **elementary substructure** of $\mathfrak{B}$, and $\mathfrak{B}$ is an **elementary extension** of $\mathfrak{A}$, and we write[1]

$$\mathfrak{A} \preccurlyeq \mathfrak{B}.$$

Then $\preccurlyeq$ is also a reflexive ordering of $\mathrm{Mod}_{\mathcal{L}}(\varnothing)$.

Now suppose $h\colon \mathfrak{A} \xrightarrow{\cong} \mathfrak{C}$. If also $\mathfrak{C} \subseteq \mathfrak{B}$, then h is an **embedding** of $\mathfrak{A}$ in $\mathfrak{B}$, and we may write

$$h\colon \mathfrak{A} \longrightarrow \mathfrak{B};$$

if in addition $\mathfrak{C} \preccurlyeq \mathfrak{B}$, then h is an **elementary embedding** of $\mathfrak{A}$ in $\mathfrak{B}$, and we may write

$$h\colon \mathfrak{A} \xrightarrow{\equiv} \mathfrak{B}.$$

The notion of *homomorphism* defined in §1.7 is weaker than embedding and will no longer be of much interest.

5.3. Implications

The **(Robinson) diagram** of $\mathfrak{A}$ is the set of *open* sentences of $\mathrm{Th}(\mathfrak{A}_A)$; it can be denoted by

$$\mathrm{diag}(\mathfrak{A}).$$

Then $\mathrm{Th}(\mathfrak{A}_A)$ may be called the **complete** or the **elementary diagram** of $\mathfrak{A}$.

Isomorphic structures are practically the same. We have already used this implicitly, in Theorem 1.4.3 for example. The following makes this precise.

LEMMA 5.3.1 (Diagram Lemma). *Suppose $h\colon A \to B$, and $\mathfrak{B}^*$ is the expansion of $\mathfrak{B}$ to $\mathcal{L}(A)$ such that*

$$a^{\mathfrak{B}^*} = h(a) \tag{†}$$

for all a in A. Then

$$\mathfrak{B}^* \models \mathrm{diag}(\mathfrak{A}) \iff h\colon \mathfrak{A} \longrightarrow \mathfrak{B}; \tag{‡}$$

$$\mathfrak{B}^* \models \mathrm{Th}(\mathfrak{A}_A) \iff h\colon \mathfrak{A} \xrightarrow{\equiv} \mathfrak{B}. \tag{§}$$

Hence if T is a theory admitting quantifier-elimination, then all embeddings of models of T are elementary embeddings. If h is surjective, then

$$\mathfrak{B}^* \models \mathrm{diag}(\mathfrak{A}) \iff h\colon \mathfrak{A} \xrightarrow{\cong} \mathfrak{B} \iff \mathfrak{B}^* \models \mathrm{Th}(\mathfrak{A}_A). \tag{¶}$$

If $A \subseteq B$, then

$$\mathfrak{B} \models \mathrm{diag}(\mathfrak{A}) \iff \mathfrak{A} \subseteq \mathfrak{B};$$

$$\mathfrak{B} \models \mathrm{Th}(\mathfrak{A}_A) \iff \mathfrak{A} \preccurlyeq \mathfrak{B}.$$

PROOF. Suppose $h\colon A \to B$. Then

$$\mathfrak{B}^* \models \varphi(\vec{a}\,) \iff \mathfrak{B}_{h[A]} \models \varphi(h(\vec{a}\,)). \tag{‖}$$

Assume first $\mathfrak{B}^* \models \mathrm{diag}(\mathfrak{A})$. We want to show $h\colon \mathfrak{A} \to \mathfrak{B}$, that is,

(1) $h(c^{\mathfrak{A}}) = c^{\mathfrak{B}}$ for all constants c in $\mathcal{L}$,
(2) $h \circ f^{\mathfrak{A}} = f^{\mathfrak{B}} \circ h$ for all function-symbols f in $\mathcal{L}$,
(3) $h[R^{\mathfrak{A}}] = h[A] \cap R^{\mathfrak{B}}$ for all predicates in $\mathcal{L}$.

[1]Some people just write $\mathfrak{A} \prec \mathfrak{B}$.

This follows by considering the open formulas $c = \mathsf{x}_0$, $f\mathsf{x}_0 \cdots \mathsf{x}_{n-1} = \mathsf{x}_n$, and $R\mathsf{x}_0 \ldots \mathsf{x}_{n-1}$. Indeed, if $c^{\mathfrak{A}} = a$, then $\operatorname{diag}(\mathfrak{A})$ contains $c = a$, so $c^{\mathfrak{B}} = h(a)$ by ($\|$). The remaining cases are similar.

Now assume conversely $h\colon \mathfrak{A} \to \mathfrak{B}$. We establish $\mathfrak{B}^* \vDash \operatorname{diag}(\mathfrak{A})$ by showing

$$\vec{a} \in \varphi^{\mathfrak{A}} \iff h(\vec{a}\,) \in \varphi^{\mathfrak{B}} \tag{$**$}$$

for all $\vec{a}$ from A, for all open formulas φ of $\mathcal{L}$. We first establish by induction that

$$h \circ t^{\mathfrak{A}} = t^{\mathfrak{B}} \circ h \tag{$\dagger\dagger$}$$

for all terms t of $\mathcal{L}$. The claim is true by definition of embedding if t is a constant or variable. If ($\dagger\dagger$) is true when $t \in \{u_0, \ldots, u_{n-1}\}$, and now t is $fu_0 \cdots u_{n-1}$, then

$$\begin{aligned}
h \circ t^{\mathfrak{A}} &= h \circ f^{\mathfrak{A}} \circ ({u_0}^{\mathfrak{A}}, \ldots, {u_{n-1}}^{\mathfrak{A}}) && [\text{by def'n of } t^{\mathfrak{A}}]\\
&= f^{\mathfrak{B}} \circ h \circ ({u_0}^{\mathfrak{A}}, \ldots, {u_{n-1}}^{\mathfrak{A}}) && [\text{by def'n of embedding}]\\
&= f^{\mathfrak{B}} \circ (h \circ {u_0}^{\mathfrak{A}}, \ldots, h \circ {u_{n-1}}^{\mathfrak{A}}) && [\text{by } (*)]\\
&= f^{\mathfrak{B}} \circ ({u_0}^{\mathfrak{B}} \circ h, \ldots, {u_{n-1}}^{\mathfrak{B}} \circ h) && [\text{by inductive hyp.}]\\
&= f^{\mathfrak{B}} \circ ({u_0}^{\mathfrak{B}}, \ldots, {u_{n-1}}^{\mathfrak{B}}) \circ h\\
&= t^{\mathfrak{B}} \circ h. && [\text{by def'n of } t^{\mathfrak{A}}]
\end{aligned}$$

Therefore ($\dagger\dagger$) holds for all t. To prove ($**$) for all $\vec{a}$, for all open formulas φ, we again use induction. If φ is $t_0 = t_1$ for some terms t_i, then

$$\begin{aligned}
\vec{a} \in \varphi^{\mathfrak{A}} &\iff {t_0}^{\mathfrak{A}}(\vec{a}\,) = {t_1}^{\mathfrak{A}}(\vec{a}\,) && [\text{by definition of } \varphi^{\mathfrak{A}}]\\
&\iff h({t_0}^{\mathfrak{A}}(\vec{a}\,)) = h({t_1}^{\mathfrak{A}}(\vec{a}\,)) && [\text{since } h \text{ is injective}]\\
&\iff {t_0}^{\mathfrak{B}}(h(\vec{a}\,))) = {t_1}^{\mathfrak{B}}(h(\vec{a}\,))) && [\text{by } (\dagger\dagger)]\\
&\iff h(\vec{a}\,) \in \varphi^{\mathfrak{B}}. && [\text{by definition of } \varphi^{\mathfrak{B}}]
\end{aligned}$$

If φ is $Rt_0 \cdots t_{n-1}$ for some terms t_i and predicate R, then:

$$\begin{aligned}
\vec{a} \in \varphi^{\mathfrak{A}} &\iff ({t_0}^{\mathfrak{A}}(\vec{a}\,), \ldots, {t_{n-1}}^{\mathfrak{A}}(\vec{a}\,)) \in R^{\mathfrak{A}} && [\text{by def'n of } \varphi^{\mathfrak{A}}]\\
&\iff h({t_0}^{\mathfrak{A}}(\vec{a}\,), \ldots, {t_{n-1}}^{\mathfrak{A}}(\vec{a}\,)) \in R^{\mathfrak{B}} && [\text{by def'n of embedding}]\\
&\iff ({t_0}^{\mathfrak{B}}(h(\vec{a}\,)), \ldots, {t_{n-1}}^{\mathfrak{B}}(h(\vec{a}\,))) \in R^{\mathfrak{B}} && [\text{by } (\dagger\dagger)]\\
&\iff h(\vec{a}\,) \in \varphi^{\mathfrak{B}}. && [\text{by def'n of } \varphi^{\mathfrak{B}}]
\end{aligned}$$

If ($**$) holds for all $\vec{a}$ when φ is ψ, and now φ is $\neg\psi$, then:

$$\begin{aligned}
\vec{a} \in \varphi^{\mathfrak{A}} &\iff \vec{a} \notin \psi^{\mathfrak{A}} && [\text{by def'n of } \varphi^{\mathfrak{A}}]\\
&\iff h(\vec{a}\,) \notin \psi^{\mathfrak{B}} && [\text{by inductive hypothesis}]\\
&\iff h(\vec{a}\,) \in \varphi^{\mathfrak{B}}. && [\text{by def'n of } \varphi^{\mathfrak{B}}]
\end{aligned}$$

Similarly, if ($**$) holds for all $\vec{a}$ when φ is χ or ψ, and now φ is $\chi \mathbin{\&} \psi$, then:

$$\begin{aligned}
\vec{a} \in \varphi^{\mathfrak{A}} &\iff \vec{a} \in \chi^{\mathfrak{A}} \;\&\; \vec{a} \in \psi^{\mathfrak{A}} && [\text{by def'n of } \varphi^{\mathfrak{A}}]\\
&\iff h(\vec{a}\,) \in \chi^{\mathfrak{B}} \;\&\; h(\vec{a}\,) \in \psi^{\mathfrak{B}} && [\text{by inductive hypothesis}]\\
&\iff h(\vec{a}\,) \in \varphi^{\mathfrak{B}}. && [\text{by def'n of } \varphi^{\mathfrak{B}}]
\end{aligned}$$

We have now established ($\ddagger$).

If $h\colon \mathfrak{A} \to \mathfrak{B}$ and h is surjective, then the foregoing proof serves to establish $(**)$ for all $\vec{a}$ and all formulas φ, once we add one more step. Suppose $(**)$ holds when φ is an $(m+1)$-ary formula ψ, and now φ is the m-ary $\exists \mathsf{x}_m\ \psi$. We have

$$\begin{aligned} \vec{a} \in \varphi^{\mathfrak{A}} &\iff (\vec{a}, b) \in \psi^{\mathfrak{A}} \text{ for some } b \text{ in } A \\ &\iff (h(\vec{a}), h(b)) \in \psi^{\mathfrak{B}} \text{ for some } b \text{ in } A \\ &\iff (h(\vec{a}), c) \in \psi^{\mathfrak{B}} \text{ for some } c \text{ in } A \\ &\iff h(\vec{a}) \in \varphi^{\mathfrak{B}}. \end{aligned}$$

(Note how the surjectivity of h was used.) This gives us (¶) and then (§). □

COROLLARY. *If* $\mathfrak{A} \cong \mathfrak{B}$, *then* $\mathfrak{A} \equiv \mathfrak{B}$. □

LEMMA 5.3.2 (Tarski–Vaught Test). *Suppose* $\mathfrak{B}$ *is an* $\mathcal{L}$*-structure,* $A \subseteq B$*, and for all singulary* $\mathcal{L}(A)$ *formulas* φ*, if* $\mathfrak{B} \vDash \exists \mathsf{x}_0\ \varphi$*, then* $\mathfrak{B} \vDash \varphi(a)$ *for some* a *in* A*. Then* A *is the universe of an elementary substructure of* $\mathfrak{B}$.

PROOF. The assumption ensures that A is the universe of a substructure $\mathfrak{A}$ of $\mathfrak{B}$. We now want to show

$$\mathfrak{A}_A \vDash \varphi(\vec{a}) \iff \mathfrak{B}_A \vDash \varphi(\vec{a}),$$

for all $\vec{a}$ from A, for all $\mathcal{L}$-formulas φ. Since $\mathfrak{A} \subseteq \mathfrak{B}$, we have the claim when φ is open. Moreover, the set of φ for which the claim holds is closed under negation and conjunction. Finally, if the claim holds when φ is an $(n+1)$-ary formula ψ, then

$$\begin{aligned} \mathfrak{A}_A \vDash \exists x\ \varphi(\vec{a}, x) &\iff \mathfrak{A}_A \vDash \varphi(\vec{a}, b) \text{ for some } b \text{ in } A \\ &\iff \mathfrak{B}_A \vDash \varphi(\vec{a}, b) \text{ for some } b \text{ in } A \\ &\iff \mathfrak{B}_A \vDash \exists x\ \varphi(\vec{a}, x) \end{aligned}$$

by assumption. This completes the induction. □

5.4. Cardinalities

The **cardinality** of a structure is the cardinality of its universe. At the end of §5.2, it was noted that the same theory may have models of different cardinalities. Usually cardinals are denoted by Greek letters like κ and λ. The cardinality of $|\mathrm{Sn}_{\mathcal{L}}|$ is sometimes denoted simply by $|\mathcal{L}|$: so this is always infinite, even though $\mathcal{L}$ may be a finite signature. Also,

$$|\mathcal{L}(X)| = \max(|\mathcal{L}|, |X|, \aleph_0).$$

THEOREM 5.4.1 (Downward Löwenheim–Skolem). *Suppose* $\mathfrak{B}$ *is an* $\mathcal{L}$*-structure,* $X \subseteq B$*, and*

$$|\mathcal{L}(X)| \leqslant \kappa \leqslant |B|.$$

Then $\mathfrak{A} \preccurlyeq \mathfrak{B}$ *for some* $\mathfrak{A}$ *of cardinality* κ *such that* $X \subseteq A$.

PROOF. If $Y \subseteq B$, let Y' be a set such that $Y \subseteq Y' \subseteq B$ and, for all singulary formulas φ of $\mathcal{L}(Y)$, if $\mathfrak{B} \vDash \exists \mathsf{x}_0\ \varphi$, then $\mathfrak{B}_B \vDash \varphi(a)$ for some a in Y'. We may assume $|Y'| \leqslant |\mathcal{L}(Y)|$. Now let $X_0 = X$ and $X_{n+1} = X_n{}'$. by the Tarski–Vaught Test, $\bigcup\{X_n \colon n \in \omega\}$ is the universe of an elementary substructure $\mathfrak{A}$ of $\mathfrak{B}$ such that $X \subseteq A$. Also $|A| \leqslant |\mathcal{L}(X)|$. We may assume also $|X| = \kappa$, in which case $|A| = \kappa$. □

A theory T is called **κ-categorical** if

(1) T has a model of cardinality κ;
(2) all models of T of cardinality κ are isomorphic (to each other).

A theory is **totally categorical** if it is κ-categorical for each infinite κ. There is an easy example. In the empty signature, structures are pure sets, and isomorphisms are just bijections. Hence, if $\mathcal{L} = \varnothing$, then $\mathrm{Con}_{\mathcal{L}}(\varnothing)$ is totally categorical.

However, there are theories that are ω-categorical (that is, $\aleph_0$-categorical), but not κ-categorical for any uncountable κ. To give an example, we first note that there are sentences σ_n (where $n > 0$) in the empty signature such that, for all theories T and structures $\mathfrak{A}$ of some common signature,

$$\mathfrak{A} \vDash T \cup \{\sigma_n : n > 0\} \iff \mathfrak{A} \vDash T \;\&\; |A| \geqslant \omega.$$

Indeed, let σ_n be

$$\exists x_0 \;\cdots \exists x_{n-1} \bigwedge_{i<j<n} x_i \neq x_j.$$

Moreover, for any singulary formula φ, if $n > 1$, we can form the sentence

$$\exists x_0 \;\cdots \exists x_{n-1} \Big(\bigwedge_{i<j<n} x_i \neq x_j \;\&\; \bigwedge_{i<n} \varphi(x_i) \Big);$$

this sentence can be abbreviated by

$$\exists^{\geqslant n} x \;\; \varphi.$$

Then

$$\mathfrak{A} \vDash \exists^{\geqslant n} x \; \varphi \iff |\varphi^{\mathfrak{A}}| \geqslant n.$$

Now suppose $\mathcal{L} = \{E\}$, where E is a binary predicate, and let T be the theory of an equivalence-relation with exactly two classes, both infinite. So T has as axioms the generalizations of

$$\begin{array}{cc} x \; E \; x, & \exists x \; \exists y \; \neg(x \; E \; y), \\ x \; E \; y \Rightarrow y \; E \; x, & x \; E \; y \vee y \; E \; z \vee z \; E \; x, \\ x \; E \; y \;\&\; y \; E \; z \Rightarrow x \; E \; z, & \exists^{\geqslant n} y \; x \; E \; y. \end{array}$$

(Note that there are infinitely many axioms.) Then T is ω-categorical. However, if κ is an *uncountable* cardinal, then T is not κ-categorical. For example, there is a model in which both E-classes have size $\aleph_1$, and a model in which one class has size $\aleph_1$, the other ω.

The **continuum** is $\mathbb{R}$, whose cardinality is $|2^{\omega}|$. The **Continuum Hypothesis** is that $|2^{\omega}| = \omega_1$; but this is logically independent of the usual axioms of set-theory. In a countable signature, there are at most continuum-many non-isomorphic countable structures, because in such a structure $\mathfrak{A}$, each symbol in the signature will be interpreted as a subset of some A^n, and there are at most continuum-many of these.

For a given signature $\mathcal{L}$, the **spectrum-function** is

$$(T, \kappa) \longmapsto I(T, \kappa),$$

where T is a theory, κ is an infinite cardinal, and $I(T, \kappa)$ is the number of non-isomorphic $\mathcal{L}$-structures of cardinality κ that are models of T. If T is included in another theory, U, of $\mathcal{L}$, then

$$I(U, \kappa) \leqslant I(T, \kappa).$$

Usually we are interested in $I(T,\kappa)$ only when T is complete.

If $|\mathcal{L}| = \omega$, then a theory of $\mathcal{L}$ is also called **countable.** Let T be such, with an infinite model. By the Downward Löwenheim–Skolem Theorem, T has a countable model. Therefore

$$1 \leqslant I(T,\omega) \leqslant |2^{\omega}|. \tag{$*$}$$

Letting T be the theory of an infinite sets (in the empty signature) shows that the lower bound cannot be improved when T is complete. **Vaught's Conjecture** is that

$$I(T,\omega) < |2^{\omega}| \implies I(T,\omega) \leqslant \omega.$$

If the Continuum Hypothesis is accepted, than this implication is trivial; the Conjecture is that the implication holds even if the Continuum Hypothesis is rejected.

The upper bound of $(*)$ cannot be improved. For example, let $\mathcal{L}$ be $\{P_n \colon n \in \omega\}$, where each P_n is a singulary predicate. Let T have as axioms all sentences of the form

$$\exists x \left(\bigwedge_{i\in I} P_i x \mathbin{\&} \bigwedge_{j\in J} \neg P_j x\right),$$

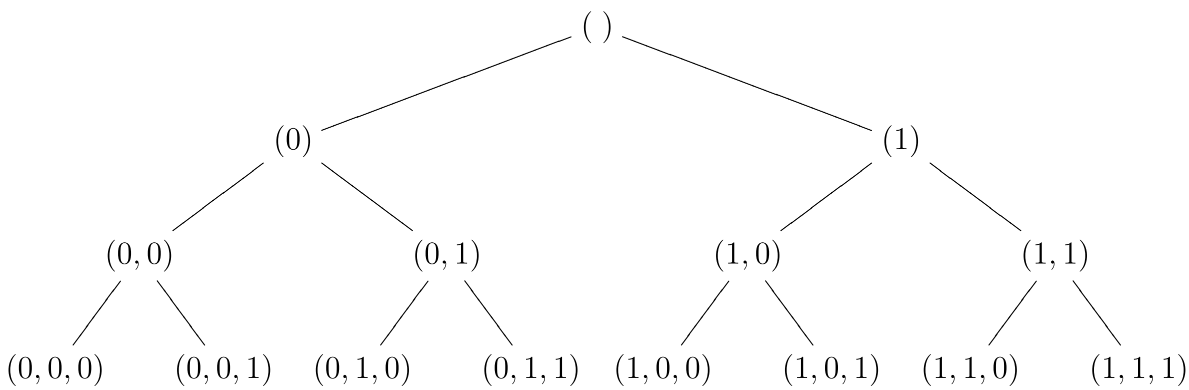
where I and J are finite disjoint subsets of ω. Then T admits quantifier-elimination and is complete (exercise). But T has continuum-many countably infinite models. To see this, we start with the uncountable model $\mathfrak{A}$, where $A = 2^{\omega}$ and

$${P_n}^{\mathfrak{A}} = \{\sigma \in A \colon s(n) = 1\}$$

for each n in ω. This has the substructure $\mathfrak{B}$, where B comprises only those elements of 2^{ω} that are eventually 0; that is, $\sigma \in B$ if and only if, for some k, if $n \geqslant k$, then $\sigma(n) = 0$. Then $\mathfrak{B}$ is a countable model of T. Indeed, if $\sigma \in B$, let $\sigma^* = \sigma \upharpoonright n+1$, where n is the greatest k such that $\sigma(k) = 1$. Then $\sigma \mapsto \sigma^*$ is an injection from B into $2^{<\omega}$, where

$$2^{<\omega} = \bigcup_{n\in\omega} 2^n.$$

This set is partially ordered by $\subset$ and is a tree, part of which can be depicted as follows.



A **branch** of a tree is a maximal totally ordered subset; the union of a branch of $2^{<\omega}$ is an element of 2^{ω}, and conversely. So $2^{<\omega}$ is countable, but has continuum-many branches. If σ and τ are distinct elements of 2^{ω}, then $\sigma(n) \neq \tau(n)$ for some n in ω, and then

$$\sigma \in {P_n}^{\mathfrak{A}} \iff \tau \notin {P_n}^{\mathfrak{A}}.$$

Hence, if also σ and τ are not in B, then $B \cup \{\sigma\}$ and $B \cup \{\tau\}$ are the universes of non-isomorphic models of T. Hence T has at least (and therefore exactly) continuum-many countable models.

When $T = \mathrm{Th}(\omega, 0, ')$, then $I(T, \omega) = \omega$. Indeed, let $\mathfrak{A} \vDash T$. There is an equivalence-relation $\sim$ on A such that $a \sim b$ if and only if $a^{(m)} = b^{(n)}$ (in the notation of the proof of Theorem 4.2.1) for some m and n in ω. Let $[a]$ be the $\sim$-class of a. If $a \sim 0^{\mathfrak{A}}$, then $([a], ')$ is isomorphic to $(\omega, ')$; if $a \not\sim 0^{\mathfrak{A}}$, then $([a], ')$ is isomorphic to $(\mathbb{Z}, x \mapsto x + 1)$. Thus $\mathfrak{A}$ is determined up to isomorphism by $|A/{\sim}|$. Moreover,

$$|A| = \begin{cases} \omega, & \text{if } |A/{\sim}| \leqslant \omega; \\ |A/{\sim}|, & \text{otherwise.} \end{cases}$$

Therefore

$$I(\mathrm{Th}(\omega, ', 0), \kappa) = \begin{cases} \omega, & \text{if } \kappa = \omega; \\ 1, & \text{if } \kappa > \omega. \end{cases}$$

Thus the theory of $(\omega, 0, ')$ is **uncountably categorical.**

We shall see in Theorem 8.1.1 that TO* is ω-categorical; however, it is not κ-categorical if $\kappa > \omega$. In a countable signature, the question of whether κ-categoricity for *one* uncountable κ implies the same for all was answered affirmatively by Michael Morley in his 1962 doctoral dissertation [**23**]. The question of which finite values can be taken by $I(T, \omega)$ is treated below in Ch. 8.

Exercises

EXERCISE 5.1. An **existential formula** is a formula of the form $\exists x_0 \cdots \exists x_{n-1}\, \varphi$, where φ is open; a **universal formula** takes the form $\forall x_0 \cdots \forall x_{n-1}\, \varphi$. Suppose $\mathfrak{A} \subseteq \mathfrak{B}$, in signature $\mathcal{L}$. Let σ be a sentence of $\mathcal{L}(A)$. Prove that,

(1) if σ is universal, and $\mathfrak{B} \vDash \sigma$, then $\mathfrak{A} \vDash \sigma$;
(2) if σ is existential, and $\mathfrak{A} \vDash \sigma$, then $\mathfrak{B} \vDash \sigma$.

EXERCISE 5.2.

(1) Show that, in the signature $\{1, {}^{-1}, \cdot\}$, every substructure of a group is a group.
(2) Show that this fails in the signature $\{1, \cdot\}$.
(3) Does it fail in the signature $\{{}^{-1}, \cdot\}$?

EXERCISE 5.3. Supply missing details in the proof of the Diagram Lemma, and prove its corollary.

EXERCISE 5.4. For any theory T, let $T_\forall$ be the set of universal consequences of T. By Exercise 5.1, every substructure of a model of T is a model of $T_\forall$. By Exercise 6.5, we shall have the converse. Meanwhile, it is possible to identify $T_\forall$ when T is the theory of fields. That is, it is possible to find a theory U such that every substructure of a field is a model of U, *and* every model of U extends to a field. Do this: find U.

EXERCISE 5.5. Axiomatize the theory of *infinite* sets and show that this theory is complete and totally categorical.

Exercise 5.6. In the signature $\{\sim, f\}$, where $\sim$ be a binary predicate, and f a singulary function-symbol, let T be the theory saying that $\sim$ is an equivalence-relation with just two classes, and f is an injective function taking every element to an inequivalent element.

(1) Find a theory U such that $T \subseteq U$ and U is totally categorical.
(2) Find a theory T^* such that every model of T extends to a model of T^* and $I(T^*, \omega) = \omega$.

Exercise 5.7. In §5.4, prove that the theory in the signature $\{P_n \colon n \in \omega\}$ is complete.

Exercise 5.8. For each n in ω, let E_n be a binary predicate. In the signature $\{E_n \colon n \in \omega\}$, let T be the theory saying that each E_n is an equivalence-relation with infinitely many classes, and each equivalence-class of E_{n+1} is included in an equivalence-class of E_n.

(1) Show that T is complete.
(2) Show that $I(T, \omega) = |2^\omega|$.

Exercise 5.9. Find $I(T, \kappa)$ for all infinite κ when T is the theory of:

(1) $(\omega, 0, {}', <)$;
(2) vector-spaces over a given scalar-field;
(3) the field $\mathbb{C}$ of complex numbers.

CHAPTER 6

Compactness

6.1. Theorem

A subset Σ of $\mathrm{Sn}_{\mathcal{L}}$ is

(1) **satisfiable** if it has a model;
(2) **finitely satisfiable** if every finite subset of Σ has a model.

We now aim to prove that every finitely satisfiable set is satisfiable: this is **compactness** for first-order logic.

To prove the Compactness Theorem for propositional logic, we used Lemma 2.6.2. The following is the same lemma for first-order logic and has the same proof.

LEMMA 6.1.1. *If Σ is finitely satisfiable, then so is $\Sigma \cup \{\sigma\}$ or $\Sigma \cup \{\neg\sigma\}$.* □

In propositional logic, we took a finitely satisfiable set Σ of propositional formulas and extended it to a set from which we could obtain a model of Σ. We can try to do something similar to prove compactness for first-order logic. Suppose Σ is a **maximal** finitely satisfiable set of first-order formulas in some signature $\mathcal{L}$: this means $\sigma \in \Sigma \iff \neg\sigma \notin \Sigma$. We can try to define an $\mathcal{L}$-structure $\mathfrak{A}$ by letting:

(1) A be the set of constants in $\mathcal{L}$;
(2) $c^{\mathfrak{A}} = c$ for every constant c in $\mathcal{L}$;
(3) $f^{\mathfrak{A}}(c_0, \dots, c_{n-1}) = d \iff (fc_0 \cdots c_{n-1} = d) \in \Sigma$;
(4) $(c_0, \dots, c_{n-1}) \in R^{\mathfrak{A}} \iff Rc_0 \cdots c_{n-1} \in \Sigma$.

We want $\mathfrak{A}$ to be a model of Σ. There are three potential problems:

(1) The signature $\mathcal{L}$ might not contain any constants.
(2) Suppose $\mathcal{L}$ does contain constants c and d. We have
$$\mathfrak{A} \models c = d \iff c^{\mathfrak{A}} = d^{\mathfrak{A}} \iff c = d.$$
So $\mathfrak{A}$ can't be a model of Σ unless either Σ does not contain the sentence $c = d$, or c and d are the same symbol.
(3) If $\mathfrak{A} \models (\neg\varphi)^x_c$ for every constant c in $\mathcal{L}$, then $\mathfrak{A} \models \forall x\ \neg\varphi$. However, possibly Σ contains all of the formulas $(\neg\varphi)^x_c$, but also $\exists x\ \varphi$.

The solutions to these problems will be as follows.

(1) We expand $\mathcal{L}$ to a signature $\mathcal{L}'$ that contains infinitely many constants. Then we enlarge Σ to a maximal finitely satisfiable subset Σ' of $\mathrm{Sn}_{\mathcal{L}'}$.
(2) Letting C be the set of constants of $\mathcal{L}'$, we define an equivalence-relation E on C by
$$c\ E\ d \iff (c = d) \in \Sigma'.$$
Then we let A be, not C, but C/E.
(3) In enlarging Σ to Σ', we ensure that, if $\exists x\ \varphi \in \Sigma'$, then $(\varphi)^x_c \in \Sigma'$ for some c in C.

The proof that these *do* solve the problems will depend on $|\mathcal{L}|$.

THEOREM 6.1.1 (Compactness). *Every finitely satisfiable set of sentences (in some signature) is satisfiable.*

PROOF. Suppose Σ is a finitely satisfiable subset of $\mathrm{Sn}_{\mathcal{L}}$. Let C be a set of new constants (so $\mathcal{L} \cap C = \varnothing$). For any $\mathcal{L}$-structure $\mathfrak{A}$, there is some a in A; so we can expand $\mathfrak{A}$ to an $\mathcal{L} \cup C$-structure $\mathfrak{A}'$ by defining

$$c^{\mathfrak{A}'} = a$$

for all c in C. In particular, Σ is still finitely satisfiable *as a set of sentences of* $\mathcal{L}'$.

Assume first that $\mathcal{L}$ is countable, and let C be countably infinite. Then we can enumerate $\mathrm{Sn}_{\mathcal{L}\cup C}$ as $\{\sigma_n : n \in \omega\}$, and C as $\{c_n : n \in \omega\}$. We shall define a chain

$$\Sigma_0 \subseteq \Sigma_1 \subseteq \Sigma_2 \subseteq \cdots,$$

where each Σ_k is finitely satisfiable, and only finitely many constants in C appear in formulas in Σ_k. The recursive definition is the following:

(1) $\Sigma_0 = \Sigma$. (By assumption, Σ_0 is finitely satisfiable, and it contains no constants of C.)

(2) Assume Σ_{2n} has been defined as required. Then define

$$\Sigma_{2n+1} = \begin{cases} \Sigma_{2n} \cup \{\sigma_n\}, & \text{if this is finitely satisfiable;} \\ \Sigma_{2n}, & \text{if not.} \end{cases}$$

Then Σ_{2n+1} is as required.

(3) Suppose Σ_{2n+1} has been defined as required. Suppose also $\sigma_n \in \Sigma_{2n+1}$, and σ_n is $\exists \mathsf{x}\ \varphi$ for some φ. The set of m such that c_m does *not* appear in a formula in Σ_{2n+1} has a least element, k. Then the set $\Sigma_{2n+1} \cup \{(\varphi)^x_{c_k}\}$ is finitely satisfiable. For, if Γ is a finite subset of Σ_{2n+1}, then it has a model $\mathfrak{A}$. Then $\mathfrak{A} \vDash (\varphi)^x_a$ for some a in A; so we can expand $\mathfrak{A}$ to a model of $\Sigma_{2n+1} \cup \{(\varphi)^x_{c_k}\}$ by interpreting c_k as a. In this case we define

$$\Sigma_{2n+2} = \Sigma_{2n+1} \cup \{(\varphi)^x_{c_k}\};$$

otherwise, let $\Sigma_{2n+2} = \Sigma_{2n+1}$. In either case, Σ_{2n+2} is as desired.

Now we define

$$\Sigma^* = \bigcup_{n\in\omega} \Sigma_n.$$

This is finitely satisfiable, since each finite subset is a subset of some Σ_n. Suppose $\Sigma^* \cup \{\sigma\}$ is finitely satisfiable. But σ is σ_n for some n, and $\Sigma_{2n} \cup \{\sigma\}$ is finitely satisfiable, so $\sigma \in \Sigma_{2n+1}$, and $\sigma \in \Sigma^*$. So Σ^* is a maximal finitely satisfiable set.

We now define a structure $\mathfrak{A}$ of $\mathcal{L} \cup C$ that will turn out to be a model of Σ. We first define

$$E = \{(c, d) \in C^2 : (c = d) \in \Sigma^*\}.$$

Then E is an equivalence-relation on C (exercise). So, we can let

$$A = C/E.$$

Let the E-class of c be denoted by $[c]$. We can define

$$c^{\mathfrak{A}} = [c].$$

If R is an n-ary predicate in $\mathcal{L}$, we let $R^{\mathfrak{A}}$ consist of those $([c_0], \dots, [c_{n-1}])$ such that Σ^* contains $Rd_0 \cdots d_{n-1}$ for some d_i such that $d_i \mathrel{E} c_i$ for each i in n. Then

$$([c_0], \dots, [c_{n-1}]) \in R^{\mathfrak{A}} \iff Rc_0 \cdots c_{n-1} \in \Sigma^*$$

(exercise). If f is an n-ary function-symbol in $\mathcal{L}$, then Σ^* contains $\exists x\ fc_0 \cdots c_{n-1} = x$ (since this sentence is true in every structure), so Σ^* contains $fc_0 \cdots c_{n-1} = d$ for some d in C, by construction of Σ^*. Moreover, if $c_i \mathrel{E} c_i'$ for each i in n, and Σ^* contains both $fc_0 \cdots c_{n-1} = d$ and $fc_0' \cdots c_{n-1}' = d'$, then $d \mathrel{E} d'$ (exercise). Hence we can define $f^{\mathfrak{A}}$ by

$$f^{\mathfrak{A}}([c_0] \cdots [c_{n-1}]) = [d] \iff fc_0 \cdots c_{n-1} = d \in \Sigma^*$$

(exercise). This works, even if f is nullary—is a constant c of $\mathcal{L}$. That is, we define

$$c^{\mathfrak{A}} = [d] \iff c = d \in \Sigma^*.$$

So we have $\mathfrak{A}$.

It remains to show $\mathfrak{A} \models \Sigma^*$. We shall do this by showing

$$\mathfrak{A} \models \sigma \iff \sigma \in \Sigma^* \tag{$*$}$$

for all sentences σ of $\mathcal{L} \cup C$. Rather, so that we can use induction, we shall show

$$\mathfrak{A} \models \varphi(\vec{c}) \iff \varphi(\vec{c}) \in \Sigma^*$$

for all $\vec{c}$ from C, for all formulas φ. We need a preliminary observation: If t is a constant term, and $c \in C$, then

$$t^{\mathfrak{A}} = [c] \iff t = c \in \Sigma^*$$

(exercise). Now suppose σ is the atomic sentence $Rt_0 \cdots t_{n-1}$, and ${t_i}^{\mathfrak{A}} = [c_i]$ for each i in n. Then

$$\begin{aligned}
\mathfrak{A} \models \sigma &\iff ({t_0}^{\mathfrak{A}}, \dots, {t_{n-1}}^{\mathfrak{A}}) \in R^{\mathfrak{A}} \\
&\iff ([c_0], \dots, [c_{n-1}]) \in R^{\mathfrak{A}} \\
&\iff Rc_0 \cdots c_{n-1} \in \Sigma^* \\
&\iff \sigma \in \Sigma^*.
\end{aligned}$$

If instead σ is the equation $t_0 = t_1$, then

$$\begin{aligned}
\mathfrak{A} \models \sigma &\iff {t_0}^{\mathfrak{A}} = {t_1}^{\mathfrak{A}} \\
&\iff [c_0] = [c_1] \\
&\iff c_0 = c_1 \in \Sigma^* \\
&\iff \sigma \in \Sigma^*.
\end{aligned}$$

Now suppose that $(*)$ holds when σ is τ. If σ is $\neg\tau$, then

$$\mathfrak{A} \models \sigma \iff \mathfrak{A} \not\models \tau \iff \tau \notin \Sigma^* \iff \sigma \in \Sigma^*$$

by maximality of Σ^*. If $(*)$ holds also when σ is θ, and now σ is $\tau \mathbin{\&} \theta$, then

$$\begin{aligned}
\mathfrak{A} \models \sigma &\iff \mathfrak{A} \models \tau \ \& \ \mathfrak{A} \models \theta \\
&\iff \tau \in \Sigma^* \ \& \ \theta \in \Sigma^* \\
&\iff \sigma \in \Sigma^*
\end{aligned}$$

by maximality of Σ^*. Finally, suppose $(*)$ holds whenever σ is $\varphi(c)$ for some c. If σ is $\exists x\ \varphi$, then

$$\begin{aligned}\mathfrak{A} \vDash \sigma &\iff \mathfrak{A} \vDash \varphi(a) \text{ for some } a \text{ in } A\\ &\iff \mathfrak{A} \vDash \varphi(c) \text{ for some } c \text{ in } C\\ &\iff \varphi(c) \in \Sigma^* \text{ for some } c \text{ in } C\\ &\iff \exists x\ \varphi \in \Sigma^*\end{aligned}$$

by definition of Σ^*. By induction, $(*)$ holds for all σ, so $\mathfrak{A} \vDash \Sigma^*$. This completes the proof when $\mathcal{L}$ is countable. If $|\mathcal{L}| = \kappa > \omega$, then we can enumerate $\mathrm{Sn}_{\mathcal{L}\cup C}$ as $\{\sigma_\alpha\colon \alpha \in \kappa\}$, and C as $\{c_\alpha\colon \alpha \in \kappa\}$. We proceed as before, only, if λ is a limit ordinal in κ, then $\Sigma_\lambda = \bigcup_{\alpha\in\lambda} \Sigma_\alpha$. □

For the model $\mathfrak{A}$ of Σ produced in the proof, we have $|A| \leqslant |C| = |\mathcal{L}|$.

6.2. Applications

THEOREM 6.2.1. *If T is a theory such that, for all n in ω, there is a model of T of size greater than n, then T has an infinite model.*

PROOF. For each n in ω, introduce a new constant c_n. Every model of the theory $T \cup \{c_i \neq c_j\colon i < j < \omega\}$ is infinite. Also this theory *has* models, by Compactness, since the theory is finitely satisfiable. Indeed, every finite subset of the theory is a subset of $T \cup \{c_i < c_j\colon i < j < n\}$ for some n. We can expand a model of T of size greater than n to a model of the larger theory by interpreting each c_i by a different element of the universe. □

For example, let **K** be the class of finite fields (considered as structures in the signature $\{+, -, \cdot, 0, 1\}$). Then $\mathrm{Th}(\mathbf{K})$ has infinite models; these are called **pseudo-finite** fields. Every field F has a **characteristic:** If

$$F \vDash \underbrace{1 + \cdots + 1}_{p} = 0$$

for some prime number p, then p is the characteristic of F, or $\mathrm{char}(F) = p$; if there is no such p, then $\mathrm{char}(F) = 0$. The field F is **perfect** if either:

(1) $\mathrm{char}(F) = 0$; or
(2) $\mathrm{char}(F) = p$ and every element of F has a p-th root.

Then perfect fields are precisely the fields that satisfy the axioms

$$\forall x\ \exists y\ (\underbrace{1 + \cdots + 1}_{p} = 0 \Rightarrow y^p = x).$$

Now, if F is finite, then $\mathrm{char}(F) = p$ for some prime p, and the function $x \mapsto x^p$ is an **automorphism** of F, that is, an isomorphism from F to itself. This shows F is perfect. Therefore the pseudo-finite fields are also perfect. In fact, axioms can be written for the theory of pseudo-finite fields [**3**].

Another field-theoretic application of Compactness is to **ordered fields,** namely, structures $\mathfrak{F}$ or $(F, +, -, \cdot, 0, 1, <)$ such that:

(1) $(F, +, -, \cdot, 0, 1)$ is a field;
(2) $(F, <)$ is a total order;
(3) $\mathfrak{F} \vDash \forall x\, \forall y\, (0 < x \mathbin{\&} 0 < y \Rightarrow 0 < x + y \mathbin{\&} 0 < x \cdot y)$;
(4) $\mathfrak{F} \vDash \forall x\, (x < 0 \Rightarrow 0 < -x)$.

An ordered field must have characteristic 0 (why?); hence $\mathbb{Q}$ can be treated as a sub-field of it. In an ordered field, the formula $0 < x$ defines the set of **positive elements.** The ordered field $\mathfrak{F}$ is **Archimedean** if, for all positive a and b in F, there is a natural number n such that

$$\mathfrak{F} \vDash a < \underbrace{b + \dots + b}_{n}.$$

Then $\mathbb{R}$ is an Archimedean ordered field. However, there is an ordered field $\mathfrak{F}$ such that $\mathfrak{F} \equiv \mathbb{R}$, but $\mathfrak{F}$ is not Archimedean. Indeed, let c be a new constant. Then the theory

$$\mathrm{Th}(\mathbb{R}) \cup \{\underbrace{1 + \dots + 1}_{n} < c \colon n \in \omega\}$$

is finitely satisfiable, since for every finite subset Σ of this theory, $\mathbb{R}$ itself expands to a model of Σ. So the theory has a model $\mathfrak{F}$, by Compactness; but this model is not Archimedean.

LEMMA 6.2.1 (Löwenheim–Skolem). *Suppose $\mathfrak{A}$ is an infinite $\mathcal{L}$-structure, and κ is an infinite cardinal such that $|\mathcal{L}| \leqslant \kappa$. Then there is an $\mathcal{L}$-structure $\mathfrak{B}$ such that $|B| = \kappa$ and $\mathfrak{A} \equiv \mathfrak{B}$.*

PROOF. Introduce κ-many new constants c_α (where $\alpha < \kappa$). In the proof of the Compactness Theorem, let Σ be $\mathrm{Th}(\mathfrak{A}) \cup \{c_\alpha \neq c_\beta \colon \alpha < \beta < \kappa\}$. This set is finitely satisfiable. Indeed, any finite subset is included in a subset $\mathrm{Th}(\mathfrak{A}) \cup \{c_{\alpha_i} \neq c_{\alpha_j} : i < j < n\}$ for some finite subset $\{\alpha_0, \dots, \alpha_{n-1}\}$ of κ. Then $\mathfrak{A}$ expands to a model of this set of sentences, once we interpret each constant c_{α_i} as a different element of A. (Since A is infinite, we can do this.) Therefore Σ is finitely satisfiable. The *proof* of Compactness now produces a model of Σ of size κ. □

THEOREM 6.2.2 (Upward Löwenheim–Skolem). *If $\mathfrak{A}$ is an infinite $\mathcal{L}$-structure, and $|\mathcal{L}(A)| \leqslant \kappa$, then $\mathfrak{A}$ has an elementary extension of cardinality κ.*

PROOF. In the lemma, replace $\mathfrak{A}$ with $\mathfrak{A}_A$ and use the Diagram Lemma. □

THEOREM 6.2.3 (Łoś–Vaught Test). *Suppose T is a satisfiable theory of $\mathcal{L}$. If*

(1) T has no finite models, and
(2) T is κ-categorical for some κ such that $|\mathcal{L}| \leqslant \kappa$,

then T is complete.

PROOF. Suppose T is satisfiable, and has no finite models, but is not complete. Then for some sentence σ, neither σ nor $\neg\sigma$ is a consequence of T. Hence, both $T \cup \{\neg\sigma\}$ and $T \cup \{\sigma\}$ have models. By Lemma 6.2.1, they have models of size κ. These models are not elementarily equivalent, so they are not isomorphic; this means T is not κ-categorical. □

Hence the theory of an equivalence-relation with just two classes, both infinite (§5.4), is ω-categorical. Likewise for many other examples given above.

Exercises

EXERCISE 6.1. Supply the missing details of the proof of Compactness.

EXERCISE 6.2.

(1) Show that every Archimedean ordered field is elementarily equivalent to some *countable, non-Archimedean* ordered field.
(2) Show that every non-Archimedean ordered field contains **infinitesimal** elements, that is, positive elements a that are less than every positive rational number.
(3) Find an explicit example of a non-Archimedean ordered field.

EXERCISE 6.3. The **order** of an element g of a group is the size of the subgroup $\{g^n : n \in \mathbb{Z}\}$ that g generates. In a **periodic group,** all elements have finite order. Suppose G is a periodic group in which there is no finite upper bound on the orders of elements. Show that $G \equiv H$ for some non-periodic group H.

EXERCISE 6.4. Suppose $(X, <)$ is an infinite total order in which X is *well-ordered* by $<$. Show that there is a total order $(X^*, <^*)$ such that

$$(X, <) \equiv (X^*, <^*),$$

but X^* is *not* well-ordered by $<^*$.

EXERCISE 6.5. For any theory T, prove that $\mathfrak{A} \models T_\forall$ if and only if $\mathfrak{A}$ is a substructure of a model of T. (See Exercise 5.4.)

EXERCISE 6.6. Find a theory that is ω-categorical, but not complete.

EXERCISE 6.7. Describe all fields F such that the theory of vector-spaces over F is complete.

EXERCISE 6.8. Give a complete axiomatization of $\mathrm{Th}(\mathbb{C})$.

CHAPTER 7

Completeness

7.1. Introduction

We aim now to establish a *sound, complete proof system* for first-order logic. The terminology is just as for propositional logic in §2.7 and §2.8. But since we shall consider various possible proof-systems $\mathcal{S}$, we shall write

$$\Sigma \vdash_{\mathcal{S}} \sigma$$

in case there is a formal proof, in the system S, of σ from Σ. If $\mathcal{T}$ is another proof system, which has the axioms and rules of inference of $\mathcal{S}$ among its own axioms and rules of inference, then we may write

$$\mathcal{S} \subseteq \mathcal{T}.$$

To the basic observations of Lemmas 2.7.2, 2.7.3, and 2.7.4, which hold quite generally, we can add

LEMMA 7.1.1.

(1) *if* $\Sigma \vdash_{\mathcal{S}} \sigma$ *and* $\mathcal{S} \subseteq \mathcal{T}$, *then* $\Sigma \vdash_{\mathcal{T}} \sigma$;
(2) *if* $\Sigma \vdash_{\mathcal{S}} \sigma$, *then* $\Sigma_0 \vdash_{\mathcal{S}} \sigma$ *for some* finite *subset* Σ_0 *of* Σ. □

7.2. Propositional logic

A generalization of Theorem 2.8.1 is

LEMMA 7.2.1. *Let* $\mathcal{S}$ *be a proof system for propositional logic. Then* $\mathcal{S}$ *is sound if and only if:*

(1) *each axiom of* $\mathcal{S}$ *is a tautology;*
(2) $\Phi \vDash \varphi$ *whenever* φ *can be inferred from* Φ *by one of the rules of inference of* $\mathcal{S}$.

PROOF. Suppose $\mathcal{S}$ is sound. If φ is an axiom of $\mathcal{S}$, then $\vdash_{\mathcal{S}} \varphi$ and therefore $\vDash \varphi$. Suppose that φ can be inferred from Φ by one of the rules of inference of $\mathcal{S}$. Then there is a subset $\{\psi_0, \dots, \psi_n\}$ of Φ for which the sequence

$$(\psi_0, \dots, \psi_{n-1}, \varphi)$$

is a deduction of φ from Φ in $\mathcal{S}$. Hence $\Phi \vdash_{\mathcal{S}} \varphi$, and therefore $\Phi \vDash \varphi$.

The converse can be proved by induction on deductions from Φ. Suppose (1) and (2) hold. Say φ has a deduction $(\psi_0, \dots, \psi_{n-1}, \varphi)$ from Φ in $\mathcal{S}$. As an inductive hypothesis, suppose $\Phi \vDash \psi_i$ for each i in n. If $\varphi \in \Phi$, then $\Phi \vDash \varphi$ trivially. If φ is an axiom of $\mathcal{S}$, then $\vDash \varphi$ by assumption, so $\Phi \vDash \varphi$. The remaining possibility is that φ can be inferred, by a rule of inference of $\mathcal{S}$, from some subset Φ_0 of $\{\psi_0, \dots, \psi_{n-1}\}$. Then $\Phi_0 \vDash \varphi$ by assumption, so $\Phi \vDash \varphi$ by Lemma 2.7.4. □

Let us return again in this chapter to using the signature $\{\neg, \Rightarrow\}$ for propositional logic. In Ch. 2 we established a sound and complete proof system in this system. Henceforth, for propositional logic, let us just use the system $\mathcal{P}$, in which all tautologies are axioms, and Deduction is the only rule of inference. This too is sound and complete.

7.3. Tautological completeness

Let $\mathcal{L}$ be a signature for first-order logic. To *prove* that a certain proof system for $\mathrm{Sn}_{\mathcal{L}}$ is complete, we shall use the method first expounded by Leon Henkin, in [**13**]. (Henkin's proof was a part of his doctoral thesis; see [**15**]. We have already used Henkin's method to prove Compactness.) The particular treatment in these notes owes something to Shoenfield's in [**31**]. I introduce the notions of *tautological* and *deductive completeness* completeness merely to make our ultimate proof system seem natural.

Let us say that a proof-system $\mathcal{S}$ for $\mathrm{Sn}_{\mathcal{L}}$ is **tautologically complete** if, from the assumption that

$$\mathbf{F}_k \in \mathrm{Con}(\mathbf{F}_0, \dots, \mathbf{F}_{k-1}) \tag{$*$}$$

where the $\mathbf{F}_i$ are n-ary propositional formulas, it follows that

$$\{\mathbf{F}_0(\vec{\sigma}\,), \dots, \mathbf{F}_{k-1}(\vec{\sigma}\,)\} \vdash_{\mathcal{S}} \mathbf{F}_k(\vec{\sigma}\,) \tag{$\dagger$}$$

for all n-tuples $\vec{\sigma}$ from $\mathrm{Sn}_{\mathcal{L}}$.

LEMMA 7.3.1. *Let $\mathcal{S}$ be a proof system for $\mathrm{Sn}_{\mathcal{L}}$. Then $\mathcal{S}$ is tautologically complete if and only if:*

(1) $\vdash_{\mathcal{S}} \sigma$ for all tautologies σ of $\mathrm{Sn}_{\mathcal{L}}$, and
(2) $\{\sigma, \sigma \Rightarrow \tau\} \vdash_{\mathcal{S}} \tau$ for all σ and τ in $\mathrm{Sn}_{\mathcal{L}}$.

PROOF. If $\mathcal{S}$ is tautologically complete, then immediately (1) follows; (2) follows since $\{P_0, P_0 \Rightarrow P_1\} \vDash P_1$.

To prove the converse, we can take advantage of the completeness of $\mathcal{P}$ and use induction in the tree of formal proofs. Suppose we have $(*)$. Then $\mathbf{F}_k$ has a formal proof from $\{\mathbf{F}_0, \dots, \mathbf{F}_{k-1}\}$. Say this proof is

$$(\mathbf{G}_0, \dots, \mathbf{G}_{m-1}, \mathbf{F}_k),$$

and suppose ($\dagger$) holds when $\mathbf{F}_k$ is any of the $\mathbf{G}_i$. There are three possibilities:

(1) If $\mathbf{F}_k \in \{\mathbf{F}_0, \dots, \mathbf{F}_{k-1}\}$, then trivially ($\dagger$) follows.
(2) If $\mathbf{F}_k$ is a tautology, then $\vdash_{\mathcal{S}} \mathbf{F}_k(\vec{\sigma}\,)$ by assumption, so ($\dagger$).
(3) If $\mathbf{G}_j$ is $(\mathbf{G}_i \Rightarrow \mathbf{F}_k)$ for some i and j in m, then, by inductive hypothesis, we have

$$\{\mathbf{F}_0(\vec{\sigma}\,), \dots, \mathbf{F}_{k-1}(\vec{\sigma}\,)\} \vdash_{\mathcal{S}} \mathbf{G}_i(\vec{\sigma}\,); \qquad \{\mathbf{F}_0(\vec{\sigma}\,), \dots, \mathbf{F}_{k-1}(\vec{\sigma}\,)\} \vdash_{\mathcal{S}} \mathbf{G}_j(\vec{\sigma}\,);$$

hence ($\dagger$) by assumption.

In all cases then, ($\dagger$) follows. □

It should be clear that a complete proof system is tautologically complete. The converse fails. For example, the proof system in which all tautologies are axioms and Detachment is the only rule of inference is not complete, since it cannot be used to prove the validity $\exists x\; x = x$.

Let $\perp$ be the negation of a tautology, say

$$\neg(\exists x\; x = x \Rightarrow \exists x\; x = x).$$

Henceforth, let $\Sigma \subseteq \mathrm{Sn}_{\mathcal{L}}$ and $\sigma \in \mathrm{Sn}_{\mathcal{L}}$.

LEMMA 7.3.2. *In a tautologically complete proof system $\mathcal{S}$, the following are equivalent:*

(1) $\Sigma \vdash_{\mathcal{S}} \neg\sigma$ *for some* σ *in* Σ*;*
(2) $\Sigma \vdash_{\mathcal{S}} \sigma$ *and* $\Sigma \vdash_{\mathcal{S}} \neg\sigma$ *for some* σ *in* $\mathrm{Sn}_{\mathcal{L}}$*;*
(3) $\Sigma \vdash_{\mathcal{S}} \sigma$ *for every* σ *in* $\mathrm{Sn}_{\mathcal{L}}$*;*
(4) $\Sigma \vdash_{\mathcal{S}} \bot$. □

If $\Sigma \vdash_{\mathcal{S}} \bot$, then Σ is **inconsistent** in $\mathcal{S}$; otherwise, it is **consistent.**

LEMMA 7.3.3. *In a complete proof system, every consistent subset of* $\mathrm{Sn}_{\mathcal{L}}$ *has a model.*

PROOF. If $\mathcal{S}$ is complete, but Σ has no model, then $\Sigma \vDash \bot$, so $\Sigma \vdash_{\mathcal{S}} \bot$ by completeness, so Σ is inconsistent. □

The converse of the lemma may fail, even if the proof system is required to be tautologically complete (exercise).

7.4. Deductive completeness

Let a proof system $\mathcal{S}$ be called **deductively complete** if $\Sigma \vdash_{\mathcal{S}} \sigma \Rightarrow \tau$ whenever $\Sigma \cup \{\sigma\} \vdash_{\mathcal{S}} \tau$.

LEMMA 7.4.1. *A tautologically and deductively complete proof system in which every consistent set has a model is complete.*

PROOF. Suppose $\mathcal{S}$ is such a system, and $\Sigma \cup \{\neg\sigma\}$ is inconsistent in $\mathcal{S}$. Then $\Sigma \cup \{\neg\sigma\} \vdash_{\mathcal{S}} \sigma$ by Lemma 7.3.2, so $\Sigma \vdash_{\mathcal{S}} \neg\sigma \Rightarrow \sigma$ by deductive completeness. But $(\neg\sigma \Rightarrow \sigma) \Rightarrow \sigma$ is a tautology, so $\Sigma \vdash_{\mathcal{S}} \sigma$ by tautological completeness.

Therefore, if $\Sigma \nvdash_{\mathcal{S}} \sigma$, then $\Sigma \cup \{\neg\sigma\}$ is consistent, so it has a model by assumption; this shows $\Sigma \nvDash \sigma$. □

LEMMA 7.4.2. *A tautologically complete proof system whose only rule of inference is Detachment is deductively complete.* □

LEMMA 7.4.3. *Suppose* Σ *is consistent in a tautologically and deductively complete proof system. The following are equivalent:*

(1) If $\Sigma \subseteq \Gamma \subseteq \mathrm{Sn}_{\mathcal{L}}$ *and* Γ *is consistent, then* $\Gamma = \Sigma$.
(2) $\neg\sigma \in \Sigma \iff \sigma \notin \Sigma$ *for all* σ *in* $\mathrm{Sn}_{\mathcal{L}}$. □

A set Σ meeting one of the conditions in the lemma can be called **maximally consistent.**

7.5. Completeness

By Lemma 7.3.1, we know of one tautologically complete proof system, namely, the system whose axioms are the tautologies, and whose rule of inference is Detachment. Let $\mathcal{S}$ be this system. Then $\mathcal{S}$ is deductively complete, by Lemma 7.4.2, and is sound, by Lemma 7.2.1. Moreover, soundness and deductive completeness are preserved if we add new valid axioms to $\mathcal{S}$. Now we shall see which valid axioms we can add in order to ensure that every consistent set has a model; then we shall have a complete system by Lemma 7.4.1.

Assuming $\mathcal{S}'$ is obtained from $\mathcal{S}$ by adding valid axioms, we try to follow the proof of the Compactness Theorem, replacing 'finitely satisfiable' with 'consistent in $\mathcal{S}'$.' Assume that $\mathcal{L}$ is countable. Suppose Σ is a consistent subset of $\mathrm{Sn}_{\mathcal{L}}$. We introduce a countably infinite set C of new constants and enumerate $\mathrm{Sn}_{\mathcal{L}\cup C}$ as $\{\sigma_n : n \in \omega\}$. We construct a chain

$$\Sigma = \Sigma_0 \subseteq \Sigma_1 \subseteq \Sigma_2 \subseteq \cdots$$

where

$$\Sigma_{2n+1} = \begin{cases} \Sigma_{2n} \cup \{\sigma_n\}, & \text{if this is consistent;} \\ \Sigma_{2n}, & \text{otherwise.} \end{cases}$$

If σ_n is $\exists x\ \varphi$, and this is in Σ_{2n+1}, then we want to define Σ_{2n+2} as

$$\Sigma_{2n+1} \cup \{\varphi^x_c\},$$

where c is a constant not used in Σ_{2n+1}. But we need to know that this set is consistent. For this, we assume that $\mathcal{S}'$ has, as axioms, the sentences

$$((\varphi)^x_c \Rightarrow \psi) \Rightarrow \exists x\ \varphi \Rightarrow \psi, \tag{$*$}$$

where c is a constant not appearing in ψ. Note that these axioms are valid. We now have:

Lemma 7.5.1. *If Γ is consistent in $\mathcal{S}'$ and contains $\exists x\ \varphi$, and c does not appear in Γ, then $\Gamma \cup \{(\varphi)^x_c\}$ is consistent in $\mathcal{S}'$.*

Proof. Suppose it's not. Then

$$\{\psi_0, \dots, \psi_{k-1}\} \cup \{(\varphi)^x_c\} \vdash_{\mathcal{S}'} \bot$$

for some ψ_i in Γ. By deductive completeness,

$$\vdash_{\mathcal{S}'} (\varphi)^x_c \Rightarrow \psi_0 \Rightarrow \cdots \Rightarrow \psi_{k-1} \Rightarrow \bot.$$

From $(*)$ and Detachment we have

$$\vdash_{\mathcal{S}'} \exists x\ \varphi \Rightarrow \psi_0 \Rightarrow \cdots \Rightarrow \psi_{k-1} \Rightarrow \bot.$$

Then $k+1$ applications of Detachment show

$$\Gamma \vdash_{\mathcal{S}'} \bot,$$

which contradicts the assumption that Γ is consistent. □

So now, given a consistent subset Σ of $\mathrm{Sn}_{\mathcal{L}}$, we can construct a consistent subset Σ^* of $\mathrm{Sn}_{\mathcal{L}\cup C}$ such that

(1) $\Sigma \subseteq \Sigma^*$;
(2) Σ^* is maximally consistent;
(3) if $(\exists x\ \varphi) \in \Sigma$, then $(\varphi)^x_c \in \Sigma$ for some c in C, that is, Σ^* **has witnessess**.

As in the proof of Compactness, we want to use Σ^* to define a model $\mathfrak{A}$ of itself. For the sake of defining the universe of $\mathfrak{A}$, we assume now that $\mathcal{S}'$ also has the axioms

$$c = c, \tag{$\dagger$}$$

$$c = c' \Rightarrow d = d' \Rightarrow c = d \Rightarrow c' = d', \tag{$\ddagger$}$$

where c, c', d and d' range over C. Let E be the relation

$$\{(c, d) \in C^2 : (c = d) \in \Sigma^*\}.$$

Lemma 7.5.2. *The relation E is an equivalence-relation.*

PROOF. We first show

$$\vdash_{\mathcal{S}'} c = c, \tag{§}$$

$$\vdash_{\mathcal{S}'} c = d \Rightarrow d = c, \tag{¶}$$

$$\vdash_{\mathcal{S}'} c = d \Rightarrow d = e \Rightarrow c = e \tag{‖}$$

for all constants c, d and e in C. We have (§) trivially by (†). An instance of (‡) is

$$c = d \Rightarrow c = c \Rightarrow c = c \Rightarrow d = c;$$

then (¶) follows by tautological completeness. Another instance of (‡) is

$$c = c \Rightarrow d = e \Rightarrow c = d \Rightarrow c = e;$$

then (‖) follows by tautological completeness. By its maximal consistency then, Σ^* contains $c = c$; and if Σ^* contains $c = d$ and $d = e$, then it contains $d = c$ and $c = e$. □

We define A to be C/E. We now define $R^{\mathfrak{A}}$ (for each n-ary predicate R in $\mathcal{L}$) as the set

$$\{([c_0], \cdots, [c_{n-1}]) \in A^n \colon (Rc_0 \cdots c_{n-1}) \in \Sigma^*\}.$$

Then we have

$$(Rc_0 \cdots c_{n-1}) \in \Sigma^* \implies ([c_0], \cdots, [c_{n-1}]) \in R^{\mathfrak{A}},$$

but perhaps not the converse. Possibly then both $Rc_0 \cdots c_{n-1}$ and $\neg Rc'_0 \cdots c'_{n-1}$ are in Σ^*, although $(c_k = c'_k) \in \Sigma^*$ in each case. To prevent this, as as axioms of $\mathcal{S}'$ we assume

$$c_0 = c'_0 \Rightarrow \cdots \Rightarrow c_{n-1} = c'_{n-1} \Rightarrow Rc_0 \cdots c_{n-1} \Rightarrow Rc'_0 \cdots c'_{n-1}.$$

We now have:

LEMMA 7.5.3. $([c_0], \cdots, [c_{n-1}]) \in R^{\mathfrak{A}} \iff (Rc_0 \cdots c_{n-1}) \in \Sigma^*$. □

Finally, suppose f is an n-ary function-symbol (where possibly $n = 0$, in which case f is a constant.) We want to be able to define $f^{\mathfrak{A}}$. (If $c \in C$, then $c^{\mathfrak{A}} = [c]$; but there might be constants of $\mathcal{L}$ as well.) To define $f^{\mathfrak{A}}$, we first need some lemmas, which are based on another axiom:

$$\varphi^x_t \Rightarrow \exists x\ \varphi, \tag{$**$}$$

where $\mathrm{fv}(\varphi) \subseteq \{x\}$ and t is a constant term. Let us assume that this is also an axiom of $\mathcal{S}'$. Then we have:

LEMMA 7.5.4 (Substitution). *If* $\mathrm{fv}(\varphi) \subseteq \{x\}$, *and the constant* c *does not appear in* φ, *then*

$$\vdash_{\mathcal{S}'} (\varphi)^x_c \Rightarrow (\varphi)^x_t$$

for all constant terms t.

PROOF. We have

$$\begin{aligned}
&\vdash_{\mathcal{S}'} (\neg\varphi)^x_t \Rightarrow \exists x\ \neg\varphi, && \text{[by } (**)\text{]}\\
&\vdash_{\mathcal{S}'} \neg\exists x\ \neg\varphi \Rightarrow (\varphi)^x_t, && \text{[by tautological completeness]}\\
&\vdash_{\mathcal{S}'} ((\neg\varphi)^x_c \Rightarrow \bot) \Rightarrow \exists x\ \neg\varphi \Rightarrow \bot, && \text{[by } (*)\text{]}\\
&\vdash_{\mathcal{S}'} (\varphi)^x_c \Rightarrow \neg\exists x\ \neg\varphi, && \text{[by tautological completeness]}
\end{aligned}$$

and hence $\vdash_{\mathcal{S}'} (\varphi)^x_c \Rightarrow (\varphi)^x_t$ by tautological completeness. □

LEMMA 7.5.5. $\vdash_{\mathcal{S}'} t = t$ *for all constant terms* t.

Proof. We have

$$\vdash_{\mathcal{S}'} c = c, \qquad \text{[by (†)]}$$
$$\vdash_{\mathcal{S}'} c = c \Rightarrow t = t, \qquad \text{[by the Substitution Lemma]}$$

and hence $\vdash_{\mathcal{S}'} t = t$ by tautological completeness. □

Lemma 7.5.6. $\vdash_{\mathcal{S}'} \exists x\ fc_0 \cdots c_{n-1} = x$.

Proof. We have

$$\vdash_{\mathcal{S}'} fc_0 \cdots c_{n-1} = fc_0 \cdots c_{n-1}, \qquad \text{[by the last lemma]}$$
$$\vdash_{\mathcal{S}'} fc_0 \cdots c_{n-1} = fc_0 \cdots c_{n-1} \Rightarrow \exists x\ fc_0 \cdots c_{n-1} = x, \qquad \text{[by (**)]}$$

hence $\vdash_{\mathcal{S}'} \exists x\ fc_0 \cdots c_{n-1} = x$ by tautological completeness. □

Finally, we assume as axioms of $\mathcal{S}'$ the sentences

$$c_0 = c_0' \Rightarrow \cdots \Rightarrow c_{n-1} = c_{n-1}' \Rightarrow fc_0 \cdots c_{n-1} = fc_0' \cdots c_{n-1}'. \tag{††}$$

This enables us to define $f^{\mathfrak{A}}$:

Lemma 7.5.7. *For each n-ary function-symbol f, there is an n-ary operation $f^{\mathfrak{A}}$ on A given by*

$$f^{\mathfrak{A}}([c_0], \dots, [c_{n-1}]) = [d] \iff (fc_0 \cdots c_{n-1} = d) \in \Sigma^*.$$

Proof. Since Σ^* is maximally consistent, we now have

$$(\exists x\ fc_0 \cdots c_{n-1} = x) \in \Sigma^*.$$

Since Σ^* has witnesses, we have $(fc_0 \cdots c_{n-1} = d) \in \Sigma^*$ for some constant d. This gives us a value for $f^{\mathfrak{A}}([c_0], \cdots, [c_{n-1}])$; we have to show that this value is unique. For this, it is enough to show

$$\vdash_{\mathcal{S}'} c_0 = c_0' \Rightarrow \cdots \Rightarrow c_{n-1} = c_{n-1}' \Rightarrow d = d' \Rightarrow fc_0 \cdots c_{n-1} = d \Rightarrow fc_0' \cdots c_{n-1}' = d'$$

for all c_k and c_k' and d and d' in C. By (††) and tautological completeness, it is enough to show

$$\vdash_{\mathcal{S}'} fc_0 \cdots c_{n-1} = fc_0' \cdots c_{n-1}' \Rightarrow d = d' \Rightarrow fc_0 \cdots c_{n-1} = d \Rightarrow fc_0' \cdots c_{n-1}' = d'.$$

In the axiom (‡), we may assume that c is not one of the variables c', d or d'. Then by the Substitution Lemma, we have

$$\vdash_{\mathcal{S}'} fc_0 \cdots c_{n-1} = c' \Rightarrow d = d' \Rightarrow fc_0 \cdots c_{n-1} = d \Rightarrow c' = d'.$$

We may also assume that c' is not one of the variables c_k, d or d'. Applying the Substitution Lemma again gives what we want. □

The structure $\mathfrak{A}$ is now determined and is a model of Σ, by the proof of the Compactness Theorem. In sum, what we have shown is:

THEOREM 7.5.1 (Completeness). *That proof system for* $\mathrm{Sn}_{\mathcal{L}}$ *is complete whose only rule of inference is Detachment, and whose axioms are the following:*

(1) *the tautologies;*
(2) $((\varphi)^x_c \Rightarrow \psi) \Rightarrow \exists x\, \varphi \Rightarrow \psi$*, where* c *does not appear in* ψ*;*
(3) $c = c$*;*
(4) $c = c' \Rightarrow d = d' \Rightarrow c = d \Rightarrow c' = d'$*;*
(5) $c_0 = c'_0 \Rightarrow \dots c_{n-1} = c'_{n-1} \Rightarrow Rc_0 \cdots c_{n-1} \Rightarrow Rc'_0 \cdots c'_{n-1}$*;*
(6) $\varphi^x_t \Rightarrow \exists x\, \varphi$*;*
(7) $c_0 = c'_0 \Rightarrow \cdots \Rightarrow c_{n-1} = c'_{n-1} \Rightarrow fc_0 \cdots c_{n-1} = fc'_0 \cdots c'_{n-1}$.

Here the notation is as follows:

- x *is a variable;*
- φ *is a formula such that* $\mathrm{fv}(\varphi) \subseteq \{x\}$*;*
- ψ *is a sentence;*
- t *is a constant term;*
- c*,* c'*,* c_k*,* c'_k*,* d *and* d' *are constants;*
- $n \in \omega$*;*
- R *is an* n*-ary predicate if* $n > 0$*; and*
- f *is an* n*-ary function-symbol (or a constant, if* $n = 0$*).*

Exercises

EXERCISE 7.1. Prove Lemma 7.1.1.

EXERCISE 7.2. Show that the proof system in which all tautologies are axioms and Detachment is the only rule of inference cannot be used to prove the validity $\exists x\; x = x$.

EXERCISE 7.3. Prove Lemma 7.3.2.

EXERCISE 7.4. Let the axioms of a proof system $\mathcal{S}$ be the tautologies, and let the rules of inference be Detachment, along with the rule that $\perp$ can be inferred from every finite set that has no model. (Note however that this is not really a *syntactical* rule.) Show that, in $\mathcal{S}$, all consistent sets have models, although the validity $\exists x\; x = x$ is not deducible in $\mathcal{S}$.

EXERCISE 7.5. Prove Lemma 7.4.2.

EXERCISE 7.6. Prove Lemma 7.4.3.

EXERCISE 7.7. Prove lemma 7.5.3.

EXERCISE 7.8. Prove the Compactness Theorem from the Completeness Theorem.

CHAPTER 8

Numbers of countable models

Our ultimate aim is to show that

$$I(T, \omega) \neq 2 \tag{‡‡}$$

whenever T is a countable, *complete* theory. The proof will require several interesting general results.

Note that proving (‡‡) requires T to be complete. For example, Let P be a singulary predicate, and in the signature $\{\mathcal{L}\}$, let T be axiomatized by

$$\forall x\ \forall y\ (Px \mathbin{\&} Py \Rightarrow x = y).$$

Then T has non-isomorphic countably infinite models $(\omega, \varnothing)$ and $(\omega, \{0\})$, and every countably infinite model is isomorphic to one of these.

8.1. Three models

In the signature $\{<\} \cup \{c_n : n \in \omega\}$, let T_3 be the theory axiomatized by

$$\mathrm{TO}^* \cup \{c_{n+1} < c_n : n \in \omega\}.$$

We shall see that T_3 is complete, and $I(T_3, \omega) = 3$. Let

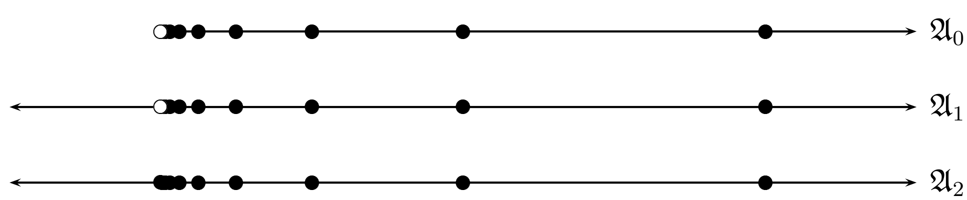
$$\begin{aligned} A_0 &= \{x \in \mathbb{Q} : 0 < x\} = \mathbb{Q} \cap (0, \infty), \\ A_1 &= \mathbb{Q} \smallsetminus \{0\}, \\ A_2 &= \mathbb{Q}. \end{aligned}$$

Then each A_k is the universe of a model $\mathfrak{A}_k$ of T_3, where $<^{\mathfrak{A}_k}$ is the usual ordering $<$, and

$$c_n{}^{\mathfrak{A}_k} = \frac{1}{2^n}.$$

Then the set $\{c_n : n \in \omega\}$ of elements has

(1) no lower bound, in $\mathfrak{A}_0$;
(2) a lower bound, but no infimum, in $\mathfrak{A}_1$;
(3) an infimum, in $\mathfrak{A}_2$.



Hence the three structures are not isomorphic. However, we shall be able to show:

(1) if $\mathfrak{B} \models T_3$ and is countable, then $\mathfrak{B} \cong \mathfrak{A}_k$ for some k in 3;
(2) T_3 is complete.

The proof of the first claim will be by the **back-and-forth method.** The following gives the prototypical example:

THEOREM 8.1.1 (Cantor [5]). TO* *is* ω*-categorical.*

PROOF. Suppose $\mathfrak{A}, \mathfrak{B} \vDash \mathrm{TO}^*$ and $|A| = \omega = |B|$. We shall show $\mathfrak{A} \cong \mathfrak{B}$. We can enumerate the universes:

$$A = \{a_n \colon n \in \omega\}, \qquad B = \{b_n \colon n \in \omega\}.$$

We shall recursively define an order-preserving bijection h from A to B. In particular, h will be $\bigcup\{h_n \colon n \in \omega\}$, where, notationally, we shall have

$$h_n = \{(a_k, b'_k) \colon k < n\} \cup \{(a'_k, b_k) \colon k < n\}.$$

We let $h_0 = \varnothing$. Suppose we have h_n so that the tuples

$$(a_0, a'_0, \dots, a_{n-1}, a'_{n-1}), \quad \text{and} \quad (b'_0, b_0, \dots, b'_{n-1}, b_{n-1})$$

have the same **order-type.** This means that, if we write these tuples as $(c_0, \dots, c_{2n-1})$ and $(c'_0, \dots, c'_{2n-1})$ respectively, then

$$c_i < c_j \iff c'_i < c'_j$$

for all i and j in $2n$. Since $\mathfrak{B}$ is a dense total order without endpoints, we can chose b'_n so that

$$(a_0, a'_0, \dots, a_{n-1}, a'_{n-1}, a_n) \quad \text{and} \quad (b'_0, b_0, \dots, b'_{n-1}, b_{n-1}, b'_n)$$

have the same order-type. Likewise, we can choose a'_n so that

$$(a_0, a'_0, \dots, a_n, a'_n), \quad \text{and} \quad (b'_0, b_0, \dots, b'_n, b_n)$$

have the same order-type. Now let $h_{n+1} = h_n \cup \{(a_n, b'_n), (a'_n, b_n)\}$. □

COROLLARY. $I(T_3, \omega) = 3$.

PROOF. Suppose $\mathfrak{B}$ is a countable model of T_3. The interpretation in $\mathfrak{B}$ of each formula

$$c_{n+1} < x \mathbin{\&} x < c_n$$

is (when equipped with the ordering induced from $\mathfrak{B}$) a countable model of TO*. The same is true for the formula $c_0 < x$. Finally, the set

$$\bigcap_{n \in \omega} \{b \in B \colon b < c_n\}$$

is one of the following:

(1) empty;
(2) a countable model of TO*;
(3) a countable dense total order with a greatest point, but no least point.

Then the previous theorem allows us to construct an isomorphism between $\mathfrak{B}$ and $\mathfrak{A}_0$, $\mathfrak{A}_1$ or $\mathfrak{A}_2$ respectively. □

The following is really a corollary of Theorem 4.1.2:

THEOREM 8.1.2. T_3 *admits elimination of quantifiers.*

PROOF. Any formula $\varphi(\vec{x})$ of $\{<, c_0, c_1, \dots\}$ can be considered as

$$\theta(\vec{x}, c_0, \dots, c_{n-1})$$

for some formula θ of $\{<\}$. By quantifier-elimination in TO^*, there is an open formula α of $\{<\}$ such that

$$\mathrm{TO}^* \models \forall \vec{x}\ \forall \vec{y}\ (\theta(\vec{x}, \vec{y}) \mathbin{\&} \bigwedge_{i<n} y_{i+1} < y_i \Leftrightarrow \alpha(\vec{x}, \vec{y})).$$

But $T_3 \models c_{i+1} < c_i$, and $T_3 \models \mathrm{TO}^*$; so

$$T_3 \models \forall \vec{x}\ (\theta(\vec{x}, \vec{c}) \Leftrightarrow \alpha(\vec{x}, \vec{c})).$$

Thus T_3 admits quantifier-elimination. □

COROLLARY. *T_3 is complete.*

PROOF. The three countable models $\mathfrak{A}_k$ form a chain:

$$\mathfrak{A}_0 \subseteq \mathfrak{A}_1 \subseteq \mathfrak{A}_2.$$

By the Diagram Lemma, the chain is elementary:

$$\mathfrak{A}_0 \preccurlyeq \mathfrak{A}_1 \preccurlyeq \mathfrak{A}_2.$$

In particular, the three structures are elementarily equivalent. Now, if $\mathfrak{B}$ is an arbitrary model of T_3, then it is infinite, so $\mathfrak{B} \equiv \mathfrak{C}$ for some countably infinite structure $\mathfrak{C}$ by the Downward Löwenheim–Skolem Theorem (5.4.1). But $\mathfrak{C} \cong \mathfrak{A}_k$ for some k, by the corollary to Cantor's Theorem. Hence $\mathfrak{B} \equiv \mathfrak{A}_0$. Thus

$$T_3 \models \mathrm{Th}(\mathfrak{A}_0);$$

so T_3 is complete. □

8.2. Omitting types

Since there is a sound, complete proof system for first-order logic, we may say that a set of sentences is **consistent** if it has a model. An **n-type** of a signature $\mathcal{L}$ is a set of n-ary formulas of $\mathcal{L}$. An n-type Φ of $\mathcal{L}$ is **realized** by $\vec{a}$ in an $\mathcal{L}$-structure $\mathfrak{A}$ if

$$\mathfrak{A} \models \varphi(\vec{a})$$

for all φ in Φ. A type not realized in a structure is **omitted** by the structure. If a consistent theory T of $\mathcal{L}$ is specified, then an **n-type of T** is an n-type Φ that is **consistent with** T: This means that Φ is realized in some model of T. Equivalently, it means that, if $\vec{c}$ is an n-tuple of new constants, then the set

$$T \cup \{\varphi(\vec{c}) \colon \varphi \in \Phi\}$$

is consistent. By Compactness, for Φ to be consistent with T, it is sufficient that

$$T \cup \{\exists \vec{x}\ \bigwedge \Phi_0\}$$

be consistent for all finite subsets Φ_0 of Φ. By Compactness also, for *any* collection of types consistent with T, there is a model of T in which all of the types are realized. An n-type Φ of T is **isolated** in T by an n-ary formula ψ if:

(1) $T \cup \{\exists \vec{x}\ \psi\}$ is consistent;
(2) $T \models \forall \vec{x}\ (\psi \Rightarrow \varphi)$ for all φ in Φ.

Hence, if ψ is satisfied by $\vec{a}$ in a model of T, then $\vec{a}$ realizes Φ. Also, if T is complete, then $T \vDash \exists \vec{x}\ \psi$, so Φ is realized in *every* model of T.

A theory is **countable** if, in its signature, only countably many formulas are inequivalent in T. It turns out that, in a *countable* theory, being isolated is the only barrier to being omitted by some model:

THEOREM 8.2.1 (Omitting Types). *Suppose T is a countable theory, and Φ is a non-isolated 1-type of T. Then Φ is omitted by some countable model of T.*

PROOF. We adjust our proof of the Compactness Theorem. As there, we introduce a set C of new constants c_n (where $n \in \omega$). We enumerate $\mathrm{Sn}_{\mathcal{L}\cup C}$ as $\{\sigma_n : n \in \omega\}$. We construct a chain

$$T = \Sigma_0 \subseteq \Sigma_1 \subseteq \cdots$$

as follows. Assume Σ_{3n} is consistent. Then let

$$\Sigma_{3n+1} = \begin{cases} \Sigma_{3n} \cup \{\sigma_n\}, & \text{if this is consistent;} \\ \Sigma_{3n}, & \text{otherwise.} \end{cases}$$

Now let

$$\Sigma_{3n+2} = \Sigma_{3n+1} \cup \{\varphi(c_k)\},$$

where k is minimal such that c_k does not appear in Σ_{3n+1}, if $\sigma_n \in \Sigma_{3n+1}$ and σ_n is $\exists x\ \varphi$; otherwise, $\Sigma_{3n+2} = \Sigma_{3n+1}$. Finally, let

$$\Sigma_{3n+3} = \Sigma_{3n+2} \cup \{\neg\psi(c_n)\},$$

where ψ is an element of Φ such that $\Sigma_{3n+2} \cup \{\neg\psi(c_n)\}$ is consistent. But we have to check that there *is* such a formula ψ in Φ. *If* there is, then we can let

$$\Sigma^* = \bigcup_{n\in\omega} \Sigma_n.$$

Then Σ^* has a countable model $\mathfrak{A}$ (as in the proof of Compactness) such that every element of A is $c^{\mathfrak{A}}$ for some c in C. But by construction, no such element can realize Φ; so $\mathfrak{A}$ omits Φ.

Now, in the definition of Σ_{3n+3}, the formula ψ exists as desired because the set $\Sigma_{3n+2} \smallsetminus T$ can be assumed to be *finite*. In particular, the formulas in this set use only finitely many constants from C. We may assume that these constants form a tuple $(c_n, \vec{d})$. Then we can write $\bigwedge \Sigma_{3n+2} \smallsetminus T$ as a sentence

$$\varphi(c_n, \vec{d}),$$

where φ is a certain formula of $\mathcal{L}$. Now, if

$$\Sigma_{3n+2} \vDash \psi(c_n)$$

for some formula ψ, then (exercise)

$$T \vDash \varphi(c_n, \vec{d}) \Rightarrow \psi(c_n),$$

and hence (exercise)

$$T \vDash \forall x\ (\exists \vec{y}\ \varphi(x, \vec{y}) \Rightarrow \psi(x)).$$

Since Φ is not isolated in T, it is not isolated by $\exists \vec{y}\ \varphi$. Therefore the set $\Sigma_{3n+2} \cup \{\neg\psi(c_n)\}$ must be consistent for some ψ in Φ. □

In the proof, it is essential that $\Sigma_n \smallsetminus T$ is finite; the proof can't be generalized to the case where T is uncountable. But the proof *can* be generalized to yield the following:

Porism. *Suppose T is a countable theory, and Φ_k is an n-type of T for some n (depending on k), for each k in ω. Then T has a countable model omitting each Φ_k.*

An n-type Φ of a theory T is called **complete** if

$$\varphi \notin \Phi \iff \neg\varphi \in \Phi$$

for all n-ary formulas φ of $\mathcal{L}$. Any n-tuple $\vec{a}$ of elements of a model $\mathfrak{A}$ of T determines a complete n-type of T, namely

$$\{\varphi \colon \mathfrak{A} \models \varphi(\vec{a}\,)\};$$

this is the **(complete) type of $\vec{a}$ in $\mathfrak{A}$** and can be denoted by

$$\mathrm{tp}_{\mathfrak{A}}(\vec{a}\,).$$

If Φ is an arbitrary n-type of T, then some $\vec{a}$ from some model $\mathfrak{A}$ of T realizes Φ, and therefore

$$\Phi \subseteq \mathrm{tp}_{\mathfrak{A}}(\vec{a}\,).$$

In particular, every type of T is included in a complete type of T.

The set of complete n-types of T can be denoted by

$$\mathrm{S}_n(T);$$

then we can let $\bigcup_{n\in\omega} \mathrm{S}_n(T)$ be denoted by

$$\mathrm{S}(T).$$

So the Omitting Types Theorem gives us that, if T is countable and $|\mathrm{S}(T)| \leqslant \omega$, then T has a countable model that omits *all* non-isolated types of T.

A structure that realizes no non-isolated types of its is called **atomic.** For example, the iterative structure $(\omega, 0, ')$ is atomic, because each of its elements k is the interpretation of the term $0^{(k)}$, so that the complete type of any tuple $(k_0, \dots, k_{n-1})$ is isolated by $\mathsf{x}_0 = 0^{(k_0)} \mathbin{\&} \cdots \mathbin{\&} \mathsf{x}_{n-1} = 0^{(k_{n-1})}$.

The order $(\omega, <)$ is atomic, because, for each element k, there is a formula φ_k such that ${\varphi_k}^{(\omega,<)} = \{k\}$ (exercise).

Theorem 8.2.2. *If $h \colon \mathfrak{A} \xrightarrow{\equiv} \mathfrak{B}$, and $\vec{a} \in A^n$, then*

$$\mathrm{tp}_{\mathfrak{B}}(h(\vec{a}\,)) = \mathrm{tp}_{\mathfrak{A}}(\vec{a}\,);$$

thus $\mathfrak{B}$ realizes all types realized in $\mathfrak{A}$.

Proof. Let $\Phi = \mathrm{tp}_{\mathfrak{A}}(\vec{a}\,)$. Then Φ is a complete type, and $\{\varphi(\vec{a}\,) \colon \varphi \in \Phi\} \subseteq \mathrm{Th}(\mathfrak{A}_A)$, so $h(\vec{a}\,)$ realizes Φ in $\mathfrak{B}$ by the Diagram Lemma. $\square$

Hence for example if h is an automorphism of $\mathfrak{A}$, then $\vec{a}$ and $h(\vec{a}\,)$ have the same complete type. For any two elements a and b of an infinite set, there is an automorphism that takes a to b. Therefore the infinite set is atomic (exercise).

However, the theory in signature $\{P_n \colon n \in \omega\}$ in §5.4 has *no* atomic models. Indeed, for every formula φ in this signature, there is a predicate P_k that does not appear in φ. Then both $\varphi \mathbin{\&} P_k\mathsf{x}_0$ and $\varphi \mathbin{\&} \neg P_k\mathsf{x}_0$ are consistent (exercise), so φ does not isolate a complete type.

8.3. Prime structures

A structure is **prime** if it embeds elementarily in every model of its theory; if that theory is T, then the structure is a **prime model of T.** (Note then that only complete theories can have prime models, simply because a prime model is elementarily equivalent to all other models.)

If T admits quantifier-elimination, then by the Diagram Lemma, all embeddings of models of T are elementary embeddings. Hence, for example, a countably infinite set is a prime model of the theory of infinite sets. Also, $(\mathbb{Q},<)$ embeds in every model of TO^*, so it is a prime model.

By the Downward Löwenheim–Skolem Theorem, a model of a countable theory T is prime, provided it embeds elementarily in all *countable* models of T. In particular then, if T is ω-categorical, then its countable model is prime.

THEOREM 8.3.1 (Vaught). *Suppose T is a countable complete theory. Then the prime models of T are precisely the countable atomic models of T.*

PROOF. Suppose $\mathfrak{A} \vDash T$.

($\Rightarrow$) If $\mathfrak{A}$ is not countable, then $\mathfrak{A}$ cannot embed in countable models of T (which must exist, by the Upward Löwenheim–Skolem Theorem, 6.2.2), so $\mathfrak{A}$ cannot be prime.

If $\mathfrak{A}$ is not atomic, then $\mathfrak{A}$ realizes some non-isolated type Φ of T. But by the Omitting-Types Theorem, T has a countable model $\mathfrak{B}$ that omits Φ. Then $\mathfrak{A}$ cannot embed elementarily in $\mathfrak{B}$, by Theorem 8.2.2.

($\Leftarrow$) Suppose $\mathfrak{A}$ is countable and atomic, and $\mathfrak{B} \vDash T$. We construct an elementary embedding of $\mathfrak{A}$ in $\mathfrak{B}$ by the back-and-forth method, except that the construction is in only one direction. Write A as $\{a_n \colon n \in \omega\}$. Then each $\mathrm{tp}_{\mathfrak{A}}(a_0,\dots,a_{n-1})$ is isolated in T by some formula φ_n. Then we have

(1) $T \vDash \exists \mathsf{x}_0 \cdots \exists \mathsf{x}_{n-1}\ \varphi_n$;
(2) $T \vDash \forall \mathsf{x}_0 \cdots \forall \mathsf{x}_{n-1}\ (\varphi_n \Rightarrow \exists \mathsf{x}_n\ \varphi_{n+1})$.

Hence we can recursively find b_k in B so that

$$\mathfrak{B} \vDash \varphi_n(b_0,\dots,b_{n-1})$$

for all n in ω. Now, every sentence in $\mathrm{Th}(\mathfrak{A}_A)$ is $\theta(a_0,\dots,a_{n-1})$ for some formula θ of $\mathcal{L}$. Then

$$T \vDash \forall \vec{\mathsf{x}}\ (\varphi_n \Rightarrow \theta),$$

so $\mathfrak{B} \vDash \theta(\vec{b})$. Therefore the function $a_k \mapsto b_k$ from A to B is an elementary embedding of $\mathfrak{A}$ in $\mathfrak{B}$. □

PORISM. *All prime models of a countable complete theory are isomorphic.*

PROOF. In the proof that $\mathfrak{A}$ embeds elementarily in $\mathfrak{B}$, if we assume also that $\mathfrak{B}$ is countable and atomic, then the full back-and-forth method gives an isomorphism between the structures. □

THEOREM 8.3.2. *Let T be a countable complete theory.*

(1) If $I(T,\omega) \leqslant \omega$, then $|\mathrm{S}(T)| \leqslant \omega$.
(2) If $|\mathrm{S}(T)| \leqslant \omega$, then T has a prime model.

PROOF. Suppose $|\mathrm{S}(T)| > \omega$. Since there are only countably many formulas, there are only countably many isolated complete types; hence there are uncountably many non-isolated complete types. Let C be an infinite set of such types. For every subset D of C, there is a countable model of T that realizes every type in D, but no type in $C \smallsetminus D$. There are uncountably many choices for D, but different choices yield non-isomorphic countable models of T. Thus $I(T, \omega) > \omega$. (In fact, $I(T, \omega) = |2^\omega|$.)

If $|\mathrm{S}(T)| \leqslant \omega$, then T has a countable atomic model by Omitting Types, hence a prime model by Theorem 8.3.1. □

Note however that $(\omega, 0, ', <)$ is a prime model of its theory T, but $I(T, \omega) = |2^\omega|$.

8.4. Saturated structures

A **saturated structure** is the opposite of an atomic structure. Atomic structures realize as *few* types as possible. Saturated structures realize as *many* types as possible; moreover, these types are allowed to have parameters from the structure.

To be precise, let $\mathfrak{M}$ be an infinite $\mathcal{L}$-structure, and let $A \subseteq M$. In this context, the set $\mathrm{S}_n(\mathrm{Th}(\mathfrak{M}_A))$ can be denoted by

$$\mathrm{S}_n(A).$$

Consider the special case where A is M itself. The set $\mathrm{S}_1(M)$, for example, contains types that include the type

$$\{x \neq a \colon a \in M\}.$$

These types cannot be realized in $\mathfrak{M}$. So we say that $\mathfrak{M}$ is **saturated**, provided that, whenever $A \subseteq M$ and $|A| < |M|$, each type in $\mathrm{S}(A)$ is realized in $\mathfrak{M}$. (In particular, if $\mathfrak{M}$ is countable here, then the sets A should be finite.)

We can construct saturated models by means of **chains,** namely, sequences $(\mathfrak{A}_n \colon n \in \omega)$ of structures such that $\mathfrak{A}_n \subseteq \mathfrak{A}_{n+1}$. Since $\subseteq$ is transitive, we may write the chain as

$$\mathfrak{A}_0 \subseteq \mathfrak{A}_1 \subseteq \mathfrak{A}_2 \subseteq \cdots .$$

If each inclusion is elementary, then the chain is called **elementary;** since the relation $\preccurlyeq$ is transitive (exercise), we may write the elementary chain as

$$\mathfrak{A}_0 \preccurlyeq \mathfrak{A}_1 \preccurlyeq \mathfrak{A}_2 \preccurlyeq \cdots .$$

The union of a chain is a structure of which all the links in the chain are substructures.

THEOREM 8.4.1 (Tarski–Vaught). *The union of an elementary chain is an elementary extension of all of the links.* □

THEOREM 8.4.2. *Suppose T is countable and complete, and $|\mathrm{S}(T)| \leqslant \omega$. Then T has a countable saturated model.*

PROOF. Suppose $\mathfrak{M}$ is a countable model of T. If A is a finite subset $\{a_k \colon k < n\}$ of M, then each element of $\mathrm{S}_m(A)$ is

$$\{\varphi(\mathsf{x}_0, \dots, \mathsf{x}_{m-1}, a_0, \dots, a_{n-1}) \colon \varphi \in p\}$$

for some p in $\mathrm{S}_{m+n}(T)$. Hence $|\mathrm{S}(A)|$ is countable. Therefore the set

$$\bigcup\{\mathrm{S}(A) \colon A \text{ is a finite subset of } M\}$$

is countable. So all of the types in this set are realized in a countable elementary extension $\mathfrak{M}'$ of $\mathfrak{M}$. Thus, if $\mathfrak{M}_0$ is a countable model of T, then we can form an elementary chain

$$\mathfrak{M}_0 \preccurlyeq \mathfrak{M}_1 \preccurlyeq \mathfrak{M}_2 \preccurlyeq \cdots$$

where $\mathfrak{M}_{n+1} = \mathfrak{M}_n{}'$. Every finite subset of N is a subset of some $\mathfrak{M}_n$, and so the types of $\mathrm{S}(A)$ are realized in $\mathfrak{M}_{n+1}$, hence in $\mathfrak{N}$. So $\mathfrak{N}$ is saturated. □

If A is a finite subset $\{a_k : k < n\}$ of M, and $\vec{a}$ is $(a_0, \dots, a_{n-1})$, we can denote $\mathfrak{M}_A$ by

$$(\mathfrak{M}, \vec{a}\,).$$

If $\mathfrak{M}$ is countable, then $\mathfrak{M}$ is called **homogeneous** if

$$\mathrm{tp}_{\mathfrak{M}}(\vec{a}\,) = \mathrm{tp}_{\mathfrak{M}}(\vec{b}\,) \implies (\mathfrak{M}, \vec{a}\,) \cong (\mathfrak{M}, \vec{b}\,)$$

for all n-tuples $\vec{a}$ and $\vec{b}$ from M, for all n in ω.

THEOREM 8.4.3. *Countable saturated structures are homogeneous.*

PROOF. The back-and-forth method. □

8.5. One model

For the sake of stating and proving the following theorem more easily, we can use the following notation. Suppose T is a theory of $\mathcal{L}$. Then equivalence in T is an equivalence-relation on the set of n-ary formulas of $\mathcal{L}$. Let the set of corresponding equivalence-classes be denoted by

$$\mathrm{B}_n(T).$$

THEOREM 8.5.1 (Ryll-Nardzewski). *Suppose T is a countable complete theory. The following statements are equivalent:*

(1) $I(T, \omega) = 1$.
(2) All types of T are isolated.
(3) Each set $\mathrm{B}_n(T)$ is finite.
(4) Each set $\mathrm{S}_n(T)$ is finite.

PROOF. (1)⇒(2): If $\mathrm{S}(T)$ contains a non-isolated type, then it is realized in some, but not all, countable models of T, so $I(T, \omega) > 1$.

(2)⇒(1): If all types of T are isolated, then all models of T are atomic, so all *countable* models of T are prime and therefore isomorphic.

(3)⇒(4): Immediate.

(4)⇒(2)&(3): Suppose $\mathrm{S}_n(T) = \{p_0, \dots, p_{m-1}\}$. For each i and j in m, if $i \neq j$, then there is a formula φ_{ij} in $p_i \smallsetminus p_j$. Let ψ_i be the formula

$$\bigwedge_{j \in m \smallsetminus \{i\}} \varphi_{ij}.$$

Then ψ_i is in p_j if and only if $j = i$. If $\mathfrak{A} \models T$, and $\vec{a}$ is an n-tuple from A, then $\mathfrak{A}$ realizes some unique p_i, and then $\mathfrak{A} \models \psi_i(\vec{a}\,)$. Conversely, if $\mathfrak{A} \models \psi_i(\vec{a}\,)$, then $\vec{a}$ must realize p_i. Therefore ψ_i isolates p_i.

If χ is an arbitrary n-ary formula, let $I = \{i \in m : \chi \in p_i\}$. Then

$$T \models \forall \vec{x}\ (\chi \Leftrightarrow \bigvee_{i \in I} \psi_i).$$

There are only finitely many possibilities for I, so $\mathrm{B}_n(T)$ is finite.

(2)⇒(4): Suppose infinitely many complete n-types are isolated in T. Since T is countable, there must be countably many such types. Say they compose the set $\{p_k : k \in \omega\}$, and each p_k is isolated by φ_k. Then the type

$$\{\neg\varphi_k : k \in \omega\}$$

is consistent with T. It is not included in any of the p_k, so it must be included in a non-isolated type. □

COROLLARY. *If $\mathfrak{A}$ is a structure in a countable signature, and $\vec{a}$ is a tuple from A, and* $\mathrm{Th}(\mathfrak{A}, \vec{a})$ *is ω-categorical, then so is* $\mathrm{Th}(\mathfrak{A})$. □

8.6. Not two models

THEOREM 8.6.1 (Vaught). *If T is a countable complete theory, then $I(T, \omega) \neq 2$.*

PROOF. Suppose $2 \leqslant I(T, \omega) \leqslant \omega$. Then T has a prime model $\mathfrak{A}$ by Theorem 8.3.2, and a saturated model $\mathfrak{B}$, by Theorem 8.4.2; moreover, some $\vec{b}$ in $\mathfrak{B}$ must have a non-isolated complete type. Suppose $(\mathfrak{C}, \vec{c}) \equiv (\mathfrak{B}, \vec{b})$. If $\mathfrak{C} \cong \mathfrak{B}$, then $(\mathfrak{C}, \vec{c}) \cong (\mathfrak{B}, \vec{a})$ for some $\vec{a}$. But then the types of $\vec{a}$, $\vec{c}$, and $\vec{b}$ are the same, so $(\mathfrak{B}, \vec{a}) \cong (\mathfrak{B}, \vec{b})$ by Theorem 8.4.3, and therefore $(\mathfrak{C}, \vec{c}) \cong (\mathfrak{B}, \vec{b})$. Since $\mathrm{Th}(\mathfrak{B}, \vec{b})$ is not ω-categorical by the corollary to Ryll-Nardzewski's Theorem, we conclude that it has a countable model $(\mathfrak{D}, \vec{d})$ such that $\mathfrak{D} \ncong \mathfrak{B}$. Also $\mathfrak{D} \ncong \mathfrak{A}$, since $\mathfrak{D}$ realizes $\mathrm{tp}_{\mathfrak{B}}(\vec{b})$. Thus $I(T, \omega) \geqslant 3$. □

Exercises

EXERCISE 8.1. For each finite n greater than 2, find a theory T such that $I(T, \omega) = n$.

EXERCISE 8.2. Supply the missing details in the proof of the Omitting Types Theorem.

EXERCISE 8.3. Show that, for each k in ω, there is a formula φ_k such that ${\varphi_k}^{(\omega,<)} = \{k\}$.

EXERCISE 8.4. Verify that an infinite set is atomic.

EXERCISE 8.5. Let T be the theory in signature $\{P_n : n \in \omega\}$ given in §5.4. Suppose P_k does not appear in the formula φ in this signature. Show that both $\varphi \mathbin{\&} P_k\mathsf{x}_0$ and $\varphi \mathbin{\&} \neg P_k\mathsf{x}_0$ are consistent with T.

EXERCISE 8.6. Prove the Tarski–Vaught Theorem on unions of elementary chains.

EXERCISE 8.7. Prove the corollary to Ryll-Nardzewski's Theorem.

EXERCISE 8.8. Prove the theorem of Chang [**7**] and Łoś and Suszko [**21**] that a theory has $\forall\exists$ axioms (that is, axioms $\forall\vec{x}\ \exists\vec{y}\ \varphi$, where φ is open) if and only if, for all chains of models of the theory, the union is also a model. Conclude for example that the union of a chain of fields is a field.

APPENDIX A

The German script

Writing in 1993, Wilfrid Hodges [**16**, Ch. 1, p. 21] observes

> Until about a dozen years ago, most model theorists named structures in horrible Fraktur lettering. Recent writers sometimes adopt a notation according to which all structures are named M, M', M^*, $\bar{M}$, M_0, M_i or occasionally N.

For Hodges, structures are A, B, C, and so forth; he refers to their universes as **domains** and denotes these by $\mathrm{dom}(A)$ and so forth. This practice is convenient if one is using a typewriter (as in the preparation of another of Hodges's books [**17**], from 1985). In 2002, David Marker [**22**] uses 'calligraphic' letters for structures, so that M is the universe of $\mathcal{M}$. I still prefer the Fraktur letters:

$\mathfrak{A}$	$\mathfrak{B}$	$\mathfrak{C}$	$\mathfrak{D}$	$\mathfrak{E}$	$\mathfrak{F}$	$\mathfrak{G}$		$\mathfrak{a}$	$\mathfrak{b}$	$\mathfrak{c}$	$\mathfrak{d}$	$\mathfrak{e}$	$\mathfrak{f}$	$\mathfrak{g}$
$\mathfrak{H}$	$\mathfrak{I}$	$\mathfrak{J}$	$\mathfrak{K}$	$\mathfrak{L}$	$\mathfrak{M}$	$\mathfrak{N}$		$\mathfrak{h}$	$\mathfrak{i}$	$\mathfrak{j}$	$\mathfrak{k}$	$\mathfrak{l}$	$\mathfrak{m}$	$\mathfrak{n}$
$\mathfrak{O}$	$\mathfrak{P}$	$\mathfrak{Q}$	$\mathfrak{R}$	$\mathfrak{S}$	$\mathfrak{T}$	$\mathfrak{U}$		$\mathfrak{o}$	$\mathfrak{p}$	$\mathfrak{q}$	$\mathfrak{r}$	$\mathfrak{s}$	$\mathfrak{t}$	$\mathfrak{u}$
	$\mathfrak{V}$	$\mathfrak{W}$	$\mathfrak{X}$	$\mathfrak{Y}$	$\mathfrak{Z}$				$\mathfrak{v}$	$\mathfrak{w}$	$\mathfrak{x}$	$\mathfrak{y}$	$\mathfrak{z}$	

A way to write these by hand is seen in a textbook of German from 1931 [**12**]:

APPENDIX B

The Greek alphabet

capital	minuscule	transliteration	name
A	α	a	alpha
B	β	b	beta
Γ	γ	g	gamma
Δ	δ	d	delta
E	ϵ	e	epsilon
Z	ζ	z	zeta
H	η	ê	eta
Θ	θ	th	theta
I	ι	i	iota
K	κ	k	kappa
Λ	λ	l	lambda
M	μ	m	mu
N	ν	n	nu
Ξ	ξ	x	xi
O	o	o	omicron
Π	π	p	pi
P	ρ	r	rho
Σ	σ, ς	s	sigma
T	τ	t	tau
Υ	υ	y, u	upsilon
Φ	ϕ	ph	phi
X	χ	ch	chi
Ψ	ψ	ps	psi
Ω	ω	ô	omega

The following remarks pertain to *ancient* Greek. The vowels are

$$\alpha, \epsilon, \eta, \iota, o, \upsilon, \omega,$$

where η is a long ϵ, and ω is a long o; the other vowels (α, ι, υ) can be long or short. Some vowels may be given tonal accents (ά, ᾶ, ὰ). An initial vowel takes either a rough-breathing mark (as in ἁ) or a smooth-breathing mark (ἀ): the former mark is transliterated by a preceding h; the latter can be ignored:

ὑπερβολή hyperbolê *hyperbola;* ὀρθογώνιον orthogônion *rectangle.*

Likewise, ῥ is transliterated as rh:

ῥόμβος rhombos *rhombus.*

A long vowel may have an iota subscript, as in ᾗ *quâ* (see p. 9). Of the two forms of minuscule sigma, the ς appears at the ends of words; elsewhere, σ appears:

βάσις basis *base.*

APPENDIX C

The natural numbers

Definition (Addition). For each m in $\mathbb{N}$, the operation $x \mapsto m + x$ on $\mathbb{N}$ is the unique homomorphism from $(\mathbb{N}, 1, \mathrm{s})$ to $(\mathbb{N}, m^{\mathrm{s}}, \mathrm{s})$ guaranteed by the Recursion Theorem (1.4.2). That is,

$$\begin{aligned} m + 1 &= m^{\mathrm{s}}, \\ m + n^{\mathrm{s}} &= (m + n)^{\mathrm{s}}. \end{aligned} \tag{$*$}$$

Lemma C.1.

(1) $1 + n = n^{\mathrm{s}}$*;*
(2) $m^{\mathrm{s}} + n = (m + n)^{\mathrm{s}}$*.* □

Theorem C.1.

(1) $n + m = m + n$*;*
(2) $(n + m) + k = n + (m + k)$*.* □

Addition exists with the foregoing properties in every inductive structure (in the sense of §1.4). Edmund Landau [**19**] shows this implicitly. See Leon Henkin [**14**] and Alexandre Borovik [**4**] for explicit discussion.

Theorem C.2. *In any inductive structure there is an operation of addition satisfying* $(*)$ *and hence the lemma and theorem.*

Proof. Let M be the set of m in the structure for which there is a singulary operation $x \mapsto m + x$ as desired. Then $1 \in M$, since if we define $1 + x$ as x^{s}, then

$$\begin{aligned} 1 + 1 &= 1^{\mathrm{s}}, \\ 1 + n^{\mathrm{s}} &= n^{\mathrm{ss}} = (1 + n)^{\mathrm{s}}. \end{aligned}$$

Suppose $k \in M$. If we define $k^{\mathrm{s}} + x$ as $(k + x)^{\mathrm{s}}$, then

$$\begin{aligned} k^{\mathrm{s}} + 1 &= (k + 1)^{\mathrm{s}} = k^{\mathrm{ss}}, \\ k^{\mathrm{s}} + n^{\mathrm{s}} &= (k + n^{\mathrm{s}})^{\mathrm{s}} = (k + n)^{\mathrm{ss}} = (k^{\mathrm{s}} + n)^{\mathrm{s}}, \end{aligned}$$

so $k^{\mathrm{s}} \in M$. By induction, all m are in M. The earlier lemma and theorem are also proved by induction alone. □

Definition (Multiplication). For each m in $\mathbb{N}$, the operation $x \mapsto m \cdot x$ on $\mathbb{N}$ is the unique homomorphism from $(\mathbb{N}, 1, \mathrm{s})$ to $(\mathbb{N}, m, x \mapsto x + m)$ guaranteed by the Recursion Theorem (1.4.2). That is,

$$\begin{aligned} m \cdot 1 &= m, \\ m \cdot (n + 1) &= m \cdot n + m. \end{aligned} \tag{$\dagger$}$$

Lemma C.2.

(1) $1 \cdot n = n$*;*
(2) $(m + 1) \cdot n = m \cdot n + n$*.* □

THEOREM C.3. *For all n, m, and k in $\mathbb{N}$,*

(1) $n \cdot m = m \cdot n$;
(2) $n \cdot (m + k) = n \cdot m + n \cdot k$;
(3) $(n \cdot m) \cdot k = n \cdot (m \cdot k)$; □

As before, only induction has been required so far.

THEOREM C.4. *In any inductive structure (in the sense of §1.4) there is an operation of multiplication satisfying* (†) *and hence the lemma and theorem.* □

The next theorem *does* need recursion.

THEOREM C.5 (Cancellation).

(1) if $n + k = m + k$, *then* $n = m$;
(2) if $n \cdot m = 1$, *then* $n = 1$ *and* $m = 1$;
(3) if $n \cdot k = m \cdot k$, *then* $n = m$. □

DEFINITION (Exponentiation). For each m in $\mathbb{N}$, the operation $x \mapsto m^x$ on $\mathbb{N}$ is the unique homomorphism from $(\mathbb{N}, 1, \mathrm{s})$ to $(\mathbb{N}, m, x \mapsto x \cdot m)$ guaranteed by the Recursion Theorem (1.4.2). That is,

$$\begin{aligned} m^1 &= m, \\ m^{n+1} &= m^n \cdot m. \end{aligned} \tag{‡}$$

THEOREM C.6.

(1) $1^n = 1$;
(2) $n^{m+k} = n^m \cdot n^k$;
(3) $(n \cdot m)^k = n^k \cdot m^k$;
(4) $(n^m)^k = n^{m \cdot k}$. □

Exponentiation requires more than induction, because of the following theorem. (See Don Zagier [**34**] for a different formulation.)

THEOREM C.7. *Let $n \in \mathbb{N}$. On the cyclic group $\mathbb{Z}/n\mathbb{Z}$ there is an operation of exponentiation satisfying* (‡) *if and only if* $n \in \{1, 2, 6, 42, 1806\}$.

PROOF. The proof is an exercise in number theory, but it involves an interesting recursive definition. We always have exponentiation as a function from $\mathbb{Z}/n\mathbb{Z} \times \mathbb{N}$ to $\mathbb{Z}/n\mathbb{Z}$. We want to find those n such that

$$x^{n+1} \equiv x \pmod{n} \tag{§}$$

for all integers x, or just all x in $\{1, \dots, n\}$. If $p^2 \mid n$ for some prime p, and $x = n/p$, then $x^k \equiv 0 \pmod{n}$ when $k > 1$, so (§) fails. So we may assume n is squarefree. It is now equivalent to ensure

$$x^{n+1} \equiv x \pmod{p}$$

for all prime factors p of n. We have this when $p \mid x$; and in the other case, it is equivalent to ensure

$$x^n \equiv 1 \pmod{p}.$$

Since x can be chosen as a primitive root of p, it is equivalent to ensure

$$p - 1 \mid n$$

for all prime factors p of n. Then also $q-1 \mid n$ for all prime factors q of $p-1$, and so forth. Keeping in mind that n must be squarefree, let us refer to a prime p as **good** if $p-1$ is squarefree and all prime factors of $p-1$ are good. Then all prime factors of n must be good. We obtain the good primes recursively. Trivially, 2 is good. For every other good prime p, we must have 2 as a factor of $p-1$. Since $2+1=3$, which is prime, 3 is good. And $2 \cdot 3+1$ is 7, which is prime and therefore good. But $2 \cdot 7+1$ is not prime. However, $2 \cdot 3 \cdot 7+1=43$, a good prime. But there are no more possibilities:

$$2 \cdot 3 \cdot 43+1=259=7 \cdot 37;$$

$$2 \cdot 7 \cdot 43+1=603=3^2 \cdot 67;$$

$$2 \cdot 3 \cdot 7 \cdot 43+1=1807=13 \cdot 139.$$

So the set of good primes is $\{2,3,7,43\}$. In this set, we have

$$p<q \iff p \mid q-1$$

Hence the set of desired n is $\{1,2,2 \cdot 3,2 \cdot 3 \cdot 7,2 \cdot 3 \cdot 7 \cdot 43\}$, which is as claimed. □

Definition (Factorial). The operation $x \mapsto x!$ on $\mathbb{N}$ is such that

$$1!=1,$$

$$(n+1)!=(n+1) \cdot n!.$$

Its uniqueness is guaranteed by the corollary to the Recursion Theorem. Indeed, the function $x \mapsto (x, x!)$ is the unique homomorphism from $(\mathbb{N}, 1, \mathrm{s})$ into

$$(\mathbb{N} \times \mathbb{N},(1,1),(x, y) \mapsto(x+1,(x+1) \cdot y)).$$

In §1.4 we obtain the strict total ordering $<$, by which $\mathbb{N}$ is well-ordered.

Theorem C.8.

(1) $m<n \iff \exists x\ m+x=n$.
(2) $m<n \iff m+k<n+k$.
(3) $m<n \iff n \cdot k<n \cdot k$. □

Hence if $m+k=n$, then k is unique and can be denoted by $n-m$.

APPENDIX D

Syntax and semantics

The Greek etymon for *syntax,* namely *ἡ σύνταξις, -εως,* refers originally to an *arranging,* a *putting together in order,* especially of soldiers. In one passage of Plato's *Republic* [**27**, 591d], it is *wealth* that may be arranged. In that passage,[1] the character of Socrates describes the wise man:

> *Οὐκοῦν, εἶπον, καὶ τὴν ἐν τῇ τῶν χρημάτων κτήσει σύνταξίν τε καὶ συμφωνίαν; καὶ τὸν ὄγκον τοῦ πλήθους οὐκ ἐκπληττόμενος ὑπὸ τοῦ τῶν πολλῶν μακαρισμοῦ ἄπειρον αὐξήσει, ἀπέραντα κακὰ ἔχων;*
>
> And will it not also be so, I said, with the arranging and harmonizing of his possessions? He will not let himself be dazzled by the felicitations of the multitude and pile up the mass of his wealth without measure, involving himself in measureless ills, will he?[2]

The arranging implied by *σύνταξις* can also be *grammatical,* a putting together of *words.*

The source of *semantics* is the Greek adjective *σημαντικός, -ή, -όν,* meaning *significant* or *meaningful.* Related words include the verb *σημαίνω* (signify) and the noun *τὸ σημεῖον* (sign). In *On Interpretation* [**1**, 16a19, b5], Aristotle defines nouns and verbs:

> *Ὄνομα μὲν οὖν ἐστὶ φωνὴ σημαντικὴ κατὰ συνθήκην ἄνευ χρόνου, ἧς μηδὲν μέρος ἐστὶ σημαντικὸν κεχωρισμένον·*
>
> *Ῥῆμα δέ ἐστι τὸ προσσημαῖνον χρόνον, οὗ μέρος οὐδὲν σημαίνει χωρίς, καὶ ἔστιν ἀεὶ τῶν καθ' ἑτέρου λεγομένων σημεῖον.*
>
> A noun is a sound, meaningful by convention, without [grammatical] tense, of which no part separately is meaningful.
>
> A verb is [a sound] signifying a tense besides; no part of it is meaningful separately; it is always a sign of *things said of* something.

The more basic *τὸ σῆμα, -ατος,* meaning *sign, mark, token,* appears in Homer (*Iliad,* X.465–468):

> *Ὣς ἄρ' ἐφώνησεν, καὶ ἀπὸ ἕθεν ὑψόσ' ἀείρας*
> *θῆκεν ἀνὰ μυρίκην· δέελον δ' ἐπὶ σῆμά τ' ἔθηκε*
> *συμμάρψας δόνακας μυρίκης τ' ἐριθηλέας ὄζους,*
> *μὴ λάθοι αὖτις ἰόντε θοὴν διὰ νύκτα μέλαιναν.*
>
> With these words, he took the spoils and set them upon a tamarisk tree, and they make a mark at the place by pulling up reeds and gathering boughs of tamarisk, that they might not miss it as they came back through the fleeting hours of darkness.[3]

---

[1]Found with the help of the Liddell–Scott lexicon [**20**].

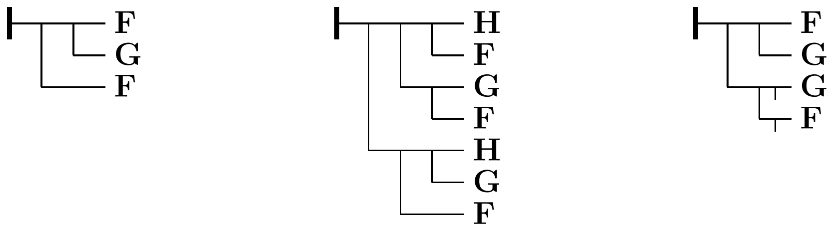
[2]The translation is adapted from Shorey's [**27**].

[3]Text and Samuel Butler's translation are from `http://www.perseus.tufts.edu`.

APPENDIX E

Syntactic entailment

What we call syntactic entailment in §2.7 seems to have its origin in the *Begriffsschrift* **[33]** of Gottlob Frege, published in 1879. (The title can be rendered as 'ideography' or 'concept writing'). In Frege's work, what we call formulas appear not as strings, but as two-dimensional figures. For example, our three axioms correspond to Frege's Judgments (1), (2), and—almost—(28); he writes them as follows:



This style of writing formulas never caught on, except in the following sense: To assert a *judgment* whose *content* is A, Frege writes

$$\boldsymbol{\vdash}\!\!\!-\!\!\!-\; A$$

The vertical bar here is the *judgment stroke,* while the horizontal is merely the *content stroke.* Frege's notation appears to be the origin of our own symbol $\vdash$.

APPENDIX F

Galois correspondences

For an arbitrary set Ω, a singulary operation $X \mapsto \mathrm{cl}(X)$ on $\mathcal{P}(\Omega)$ is a **closure-operator,** or just a **closure,** on Ω if it is:

(1) **increasing:** $A \subseteq \mathrm{cl}(A)$;
(2) **monotone:** $\mathrm{cl}(A) \subseteq \mathrm{cl}(B)$ whenever $A \subseteq B$; and
(3) **idempotent:** $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$.

The closure $X \mapsto \mathrm{cl}(X)$ is called **finitary** if

$$\mathrm{cl}(A) = \bigcup_{X \subseteq_{\mathrm{f}} A} \mathrm{cl}(X)$$

for all A in $\mathcal{P}(\Omega)$. (Here $X \subseteq_{\mathrm{f}} A$ means X is a *finite* subset of A, as on p. 36.) Examples include the following.

(1) On any set, the identity-function $X \mapsto X$ is trivially a finitary closure.
(2) On PF, the function $\mathbf{F} \mapsto \mathrm{Con}(\mathbf{F})$ is a closure, by Theorem 2.5.1; it is finitary, by the corollary to the Compactness Theorem (2.6.1).
(3) If G is a group, the function $X \mapsto \langle X\rangle$ taking a subset of G to the group that it generates is a finitary closure on G.
(4) If Ω is a topological space, the function taking a subset of Ω to its topological closure is a closure on Ω (usually not finitary).

Closures can arise from a *Galois correspondence* between two sets. Suppose A and B are sets, and R is a relation from A to B. If $C \subseteq A$, and $D \subseteq B$, let

$$C' = \bigcap_{x \in C} \{y \in B \colon x \mathrel{R} y\} = \{y \in B \colon \forall x\ (x \in C \Rightarrow x \mathrel{R} y)\}$$
$$D' = \bigcap_{y \in D} \{x \in A \colon x \mathrel{R} y\} = \{x \in A \colon \forall y\ (y \in D \Rightarrow x \mathrel{R} y)\}.$$

So we have functions $X \mapsto X'$ from $\mathcal{P}(A)$ to $\mathcal{P}(B)$ and from $\mathcal{P}(B)$ to $\mathcal{P}(A)$. These functions are inclusion-reversing; so the operations $X \mapsto X''$ on $\mathcal{P}(A)$ and $\mathcal{P}(B)$ are inclusion-preserving (monotone). Moreover,

$$\begin{aligned} C'' &= \{x \in A \colon \forall y\ (y \in C' \Rightarrow x \mathrel{R} y)\} \\ &= \{x \in A \colon \forall y\ (\forall z\ (z \in C \Rightarrow z \mathrel{R} y) \Rightarrow x \mathrel{R} y)\}, \end{aligned}$$

so

$$C \subseteq C''; \tag{¶}$$

similarly,

$$D \subseteq D''. \tag{$\|$}$$

Thus the $X \mapsto X''$ are increasing. Replacing C with D' in (¶), we get $D' \subseteq D'''$; but ($\|$) implies $D''' \subseteq D'$; therefore

$$D' = D'''.$$

Likewise, $C' = C'''$. Hence $C'' = C''''$ and $D'' = D''''$, so the $X \mapsto X''$ are idempotent and are therefore closures. Moreover, the functions $X \mapsto X'$ give bijections, not (necessarily) between $\mathcal{P}(A)$ and $\mathcal{P}(B)$, but between $\{X' \colon X \subseteq B\}$ and $\{X' \colon X \subseteq A\}$. In short, there is a **Galois correspondence** between these two sets.

The closures in the examples above are $X \mapsto X''$ on $\mathcal{P}(A)$ when

(1) A is a set, B is the same set, and R is $\neq$;
(2) A is PF, B is $\mathbb{B}^V$, and R is the converse of $\vDash$;
(3) A is G, B is the set of subgroups of G, and R is $\in$;
(4) A is the space, B is the topology (namely, the set of closed subsets), and R is $\in$.

In field-theory arises the original Galois correspondence. If L/K is a finite normal separable extension of fields, then the fields F such that $K \subseteq F \subseteq L$ are in bijection with the subgroups of $\mathrm{Aut}(L/K)$. This correspondence arises as above in case A is L, and B is $\mathrm{Aut}(L/K)$, and

$$R = \{(x, \sigma) \in L \times \mathrm{Aut}(L/K) \colon x^\sigma = x\}.$$

APPENDIX G

Definable sets

To define the interpretations of formulas in a structure recursively, we start out as in §3.2 with the interpretations of atomic formulas. We define the interpretations of negations and conjunctions as in §3.3. To deal with the existential quantifier, we might proceed as follows.

If I is a finite subset of ω, and if $\{i\colon \mathsf{x}_i \in \mathrm{fv}(\varphi)\} \subseteq I$, let us say that φ is I-ary. In this case, suppose the interpretation $\varphi^{\mathfrak{A}}$ has been defined as a subset of A^I. If $j \in \omega$, let π^I_j be the function

$$(x_i\colon i \in I) \longmapsto (x_i\colon i \in I \smallsetminus \{j\})$$

from A^I to $A^{I\smallsetminus\{j\}}$. Then we can define

$$\exists \mathsf{x}_j\ \varphi^{\mathfrak{A}} = (\pi^I_j)[\varphi^{\mathfrak{A}}].$$

Suppose J is another finite subset of ω, disjoint from I. Then φ is also $I \cup J$-ary, and the interpretation of φ as such is

$$\varphi^{\mathfrak{A}} \times A^J,$$

which can be understood as the set of functions h on $I \cup J$ such that $h \restriction I \in \varphi^I$ and $h \restriction J \in A^J$.

Without the requirement that I and J be disjoint, suppose $\alpha\colon J \to I$. We obtain the function α^* from A^I to A^J, namely

$$(x_i\colon i \in I) \longmapsto (x_{\alpha(j)}\colon j \in J).$$

In particular, if $J = I \smallsetminus \{j\}$, and α is the inclusion of this in I, then $\alpha^* = \pi^I_j$.

Now suppose again that I and J are disjoint. If $I = \{i_0, \dots, i_{m-1}\}$, and $J = \{j_0, \dots, j_{n-1}\}$, then $\alpha^*[\varphi^{\mathfrak{A}}]$ is the interpretation of the formula

$$\exists x_{i_0}\ \cdots \exists x_{i_{m-1}}\ (\varphi \mathbin{\&} x_{j_0} = x_{\alpha(j_0)} \mathbin{\&} \cdots \mathbin{\&} x_{j_{n-1}} = x_{\alpha(j_{n-1})}),$$

written more simply as

$$\exists (x_i\colon i \in I)\ (\varphi \mathbin{\&} \bigwedge_{j\in J} x_j = x_{\alpha(j)}).$$

If ψ is J-ary, then $(\alpha^*)^{-1}[\psi^{\mathfrak{A}}]$ is the interpretation of

$$\exists (x_j\colon j \in J)\ (\psi \mathbin{\&} \bigwedge_{j\in J} x_j = x_{\alpha(j)}). \tag{$**$}$$

If I and J are not disjoint, then there is still some K, disjoint from each of them, for which there are β from J to K, and γ from K to I, such that $\alpha = \gamma\circ\beta$. Then $\alpha^* = \beta^*\circ\gamma^*$, so $\alpha^*[\varphi^{\mathfrak{A}}]$ and $(\alpha^*)^{-1}[\psi^{\mathfrak{A}}]$ are still definable sets.

Note that $(**)$ might be written more simply as

$$\varphi(x_{\alpha(j)}\colon j \in J).$$

But this is not necessarily the result of substituting $x_{\alpha(j)}$ for x_j.

Bibliography

[1] Aristotle, *Categories, on interpretation, and prior analytics*, Loeb Classical Library, vol. 325, Harvard University Press and William Heinemann Ltd, Cambridge, Massachusetts and London, 1973, with an English translation by H. P. Cooke and H. Tredennick.

[2] ———, *The metaphysics, books i–ix*, Loeb Classical Library, vol. 271, Harvard University Press and William Heinemann Ltd., Cambridge, Massachusetts, and London, 1980, With an English translation by Hugh Tredennick. First printed 1933.

[3] James Ax, *The elementary theory of finite fields*, Ann. of Math. (2) **88** (1968), 239–271. MR MR0229613 (37 #5187)

[4] Alexandre V. Borovik, *Shadows of the truth: Metamathematics of elementary mathematics*, `http://www.maths.manchester.ac.uk/~avb/MEM.pdf`, August 25, 2008.

[5] Georg Cantor, *Contributions to the founding of the theory of transfinite numbers*, Cosimo Classics, New York, 2007, translated, and provided with an introduction and notes, by Philip E. B. Jourdain; originally published 1915.

[6] C. C. Chang and H. J. Keisler, *Model theory*, North-Holland Publishing Co., Amsterdam, 1973, Studies in Logic and the Foundations of Mathematics, Vol. 73. MR MR0409165 (53 #12927)

[7] Chen Chung Chang, *On unions of chains of models*, Proc. Amer. Math. Soc. **10** (1959), 120–127. MR MR0103812 (21 #2576)

[8] Alonzo Church, *Introduction to mathematical logic. Vol. I*, Princeton University Press, Princeton, N. J., 1956. MR 18,631a

[9] R. G. Collingwood, *The new Leviathan, or Man, society, civilization, and barbarism*, Oxford, 2000, revised edition, edited and introduced by David Boucher.

[10] Richard Dedekind, *Essays on the theory of numbers. I: Continuity and irrational numbers. II: The nature and meaning of numbers*, authorized translation by Wooster Woodruff Beman, Dover Publications Inc., New York, 1963. MR MR0159773 (28 #2989)

[11] Robert Avery et al. (ed.), *İngilizce–Türkçe Redhouse sözlüğü*, SEV Matbaacılık ve Yayıncılık, 1974, 33rd printing, 2002.

[12] Roe-Merrill S. Heffner, *Brief German grammar*, D. C. Heath and Company, Boston, 1931.

[13] Leon Henkin, *The completeness of the first-order functional calculus*, J. Symbolic Logic **14** (1949), 159–166. MR MR0033781 (11,487d)

[14] ———, *On mathematical induction*, Amer. Math. Monthly **67** (1960), 323–338. MR MR0120156 (22 #10913)

[15] ———, *The discovery of my completeness proofs*, Bull. Symbolic Logic **2** (1996), no. 2, 127–158. MR MR1396852 (97c:03005)

[16] Wilfrid Hodges, *Model theory*, Encyclopedia of Mathematics and its Applications, vol. 42, Cambridge University Press, Cambridge, 1993. MR 94e:03002

[17] ———, *Building models by games*, Dover Publications, Mineola, New York, 2006, original publication, 1985. MR MR812274 (87h:03045)

[18] David Hume, *A treatise of human nature*, second ed., Oxford, 1978, Edited, with an analytical index, by L. A. Selby-Bigge, with text revised and variant readings by P. H. Nidditch.

[19] Edmund Landau, *Foundations of analysis. The arithmetic of whole, rational, irrational and complex numbers*, third ed., Chelsea Publishing Company, New York, N.Y., 1966, translated by F. Steinhardt; first edition 1951; first German publication, 1929. MR 12,397m

[20] Henry George Liddell and Robert Scott, *A Greek-English lexicon*, Clarendon Press, Oxford, 1940, revised and augmented throughout by Sir Henry Stuart Jones.

[21] Jerzy Łoś and Roman Suszko, *On the extending of models (IV): Infinite sums of models*, Fund. Math. **44** (1957), 52–60. MR MR0089813 (19,724c)

[22] David Marker, *Model theory: an introduction*, Graduate Texts in Mathematics, vol. 217, Springer-Verlag, New York, 2002. MR 1 924 282

[23] Michael Morley, *Categoricity in power*, Trans. Amer. Math. Soc. **114** (1965), 514–538. MR MR0175782 (31 #58)

[24] Murray et al. (eds.), *The compact edition of the Oxford English Dictionary*, Oxford University Press, 1973.

[25] Filiz Oktem, *Uygulamalı Latin dili*, Sosyal Yayınlar, Eylül 1996.

[26] Giuseppe Peano, *The principles of arithmetic, presented by a new method*, From Frege to Gödel (Jean van Heijenoort, ed.), Harvard University Press, 1967, First published 1889.

[27] Plato, *Republic*, Loeb Classical Library, Harvard University Press and William Heinemann Ltd., Cambridge, Massachusetts, and London, 1980, with an English Translation by Paul Shorey, in two volumes.

[28] Emil L. Post, *Introduction to a general theory of elementary propositions*, Amer. J. Math. **43** (1921), no. 3, 163–185.

[29] Philipp Rothmaler, *Introduction to model theory*, Algebra, Logic and Applications, vol. 15, Gordon and Breach Science Publishers, Amsterdam, 2000, prepared by Frank Reitmaier, translated and revised from the 1995 German original by the author. MR MR1800596 (2001h:03002)

[30] J. D. Salinger, *Nine stories*, Little, Brown and Company, New York, 2001, originally published 1953.

[31] Joseph R. Shoenfield, *Mathematical logic*, Association for Symbolic Logic, Urbana, IL, 2001, reprint of the 1973 second printing. MR MR1809685 (2001h:03003)

[32] Robert R. Stoll, *Set theory and logic*, Dover Publications Inc., New York, 1979, corrected reprint of the 1963 edition. MR 83e:04002

[33] Jean van Heijenoort (ed.), *Frege and Gödel. Two fundamental texts in mathematical logic*, Harvard University Press, Cambridge, Mass., 1970. MR MR0263601 (41 #8202)

[34] Don Zagier, *Problems posed at the St Andrews Colloquium, 1996*, `http://www-groups.dcs.st-and.ac.uk/~john/Zagier/Problems.html`, accessed September 16, 2008.

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"You're just being logical," Teddy said to him impassively.

"I'm just being what?" Nicholson asked, with a little excess of politeness.

"Logical. You're just giving me a regular, intelligent answer," Teddy said. "I was trying to help you. You asked me how I get out of the finite dimensions when I feel like it. I certainly don't use logic when I do it. Logic's the first thing you have to get rid of."

Nicholson removed a flake of tobacco from his tongue with his fingers.

"You know Adam?" Teddy asked him.

"Do I know who?"

"Adam. In the Bible."

Nicholson smiled. "Not personally," he said drily.

Teddy hesitated. "Don't be angry with me," he said. "You asked me a question, and I'm—"

"I'm not *ang*ry with you, for heaven's sake."

"Okay," Teddy said. He was sitting back in his chair, but his head was turned toward Nicholson. "You know that apple Adam ate in the Garden of Eden, referred to in the Bible?" he asked. "You know what was in that apple? Logic. Logic and intellectual stuff. That was all that was in it. So—this is my point—what you have to do is vomit it up if you want to see things as they really are. I mean if you vomit it up, then you won't have any more trouble with blocks of wood and stuff. You won't see everything stopping *off* all the time. And you'll know what your arm really is, if you're interested. Do you know what I mean? Do you follow me?"

—J. D. Salinger, "Teddy" [**30**, pp. 290f.]