Completeness in first-order logic

David Pierce

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0 Preface

These notes are for Math 406 in the fall of 2004. In class, we have proved the Compactness Theorem of first-order logic. In these notes, we establish a complete proof-system for first-order logic. The result is Theorem 6.8 on p. 12. The proof of this theorem follows the pattern of our proof of Compactness.

First-order logic is based on propositional logic. It will be useful to have a general description of *logics* that encompasses both propositional and first-order logic. So, this is where we begin. All sections following § 3 concern first-order logic, unless otherwise noted.

There are a few exercises, on pp. 3, 6, 7, 7, 8, 8 and 10.

1 Logic in general

A logic has an **alphabet**, which is just a certain non-empty set; the members of this set can be called the **symbols** of the logic. These symbols can be put together to form **strings**. If we want a formal definition, we can say that such a string is a **finite, non-empty sequence** of symbols of the logic; that is, the string is a function $k \mapsto s_k$ from $\{0, 1, \ldots, n\}$ into the alphabet, for some n in ω. We usually write this function as

$$s_0 s_1 \cdots s_n;$$

this the result of **juxtaposing** the symbols s_k in the prescribed order. Such a string has **sub-strings**, namely the strings

$$s_\ell s_{\ell+1} \cdots s_m,$$

where $0 \leqslant \ell \leqslant m \leqslant n$; the sub-string is **proper** if $0 < \ell$ or $m < n$. Certain strings will be *formulas* of the logic. In particular, certain strings will be **atomic** formulas. Some **rules of construction** are specified for converting certain finite sets of strings into other strings. Then a **formula** of the logic is a member of the smallest set X of strings such that:

- (∗) all atomic formulas are in X; and
- (†) X contains every string that results from applying a rule of construction to a set of elements of X.

Hence properties of all formulas can be proved by *induction*.

Moreoever, it is required that, for every formula that is not atomic, there is exactly one rule of construction and one set of formulas such that the original formula results from applying that rule to that set. This is the principle of **uniquely readability** as formulas; it makes possible the *recursive* definition of functions on the set of formulas.

For any logic, a **proof-system** consists of:

- (∗) **axioms**, which are just certain formulas of the logic;
- (†) **rules of inference**, that is, ways of inferring certain formulas from certain *finite* sets of formulas.

So the notions of *axiom* and *rule of inference* are parallel to the notions of *atomic formula* and *rule of construction*. However, in a proof-system, there is no requirement corresponding to unique readability.

Let $\mathcal{S}$ be proof-system. A **deduction** or **formal proof in $\mathcal{S}$ of** the formula ϕ **from** the set Φ of formulas is a sequence

$$\psi_0, \ldots, \psi_n$$

of formulas where ψ_n is ϕ, and for each k such that $k \leqslant n$, one of the following holds:

- (∗) $\psi_k \in \Phi$, or
- (†) ψ_k is an axiom of $\mathcal{S}$, or
- (‡) ψ_k follows from some subset of $\{\psi_j : j < k\}$ by one of the rules of inference of $\mathcal{S}$.

To denote that such a deduction exists, we can write

$$\Phi \vdash_{\mathcal{S}} \phi.$$

Then we can say that ϕ is **deducible** from Φ in $\mathcal{S}$. In case Φ is empty, we can just write

$$\vdash_{\mathcal{S}} \phi,$$

and we can call ϕ a **theorem** of $\mathcal{S}$.

Here are some basic facts: {lem:gen}

Lemma 1.1.

(∗) *Every non-empty initial segment of a deduction is also a deduction;*

(†) *if* $\Phi \vdash_{\mathcal{S}} \phi$ *and* $\Phi \subseteq \Phi^*$, *then* $\Phi^* \vdash_{\mathcal{S}} \phi$;

(‡) *if* $\Phi \vdash_{\mathcal{S}} \phi$, *then* $\Phi_0 \vdash_{\mathcal{S}} \phi$ *for some* finite *subset* Φ_0 *of* Φ;

(§) *if* $\Phi \vdash_{\mathcal{S}} \psi$ *for each* ψ *in* Ψ, *and* $\Psi \vdash_{\mathcal{S}} \chi$, *then* $\Phi \vdash_{\mathcal{S}} \chi$.

Proof. **Exercise.** □ {ex1}

2 Propositional logic

We shall work here with the propositional logic whose alphabet consists of:

(∗) the propositional variables P_k, where $k \in \omega$;

(†) the connectives $\neg$ and $\rightarrow$;

(‡) the left bracket (and the right bracket).

The atomic formulas are then the propositional variables. There are two rules of construction:

(∗) From the string A, construct $\neg A$.

(†) From the strings A and B, construct $(A \rightarrow B)$.

Note that the same formula might be both $(A \rightarrow B)$ and $(C \rightarrow D)$ for some *strings* A, B, C and D such that A is not C. But if all of these strings are *formulas*, then (as one can prove) A must be C. We use F and G and H as syntactical variables for propositional formulas.

In propositional logic, there is a notion of **truth**, which we can develop as follows. If $S \subseteq \omega$, let 2^S be the set of functions from S to 2. We can consider 2 as the universe of the field $\mathbb{F}_2$; then a ring-structure on 2^S is induced. If F is a propositional formula, and all variables appearing in F are in S, then there is a function $\hat{F}$ from 2^S into 2, as given by the following recursive definition:

(∗) If F is P_k, then $\hat{F}(\alpha) = \alpha(k)$ for all α in 2^ω.

(†) If F is $\neg G$, then $\hat{F} = 1 + \hat{G}$.

(‡) If F is $(G \rightarrow H)$, then $\hat{F} = 1 + \hat{G} \cdot (1 + \hat{H})$.

Suppose S is the set of variables actually appearing in F, and $\hat{F}(\alpha) = 1$ for all α in 2^S; then F is called a **tautology**.

An element α of 2^ω can be called a **structure** for propositional logic. (Alternatively, the set $\{P_n : \alpha(n) = 1\}$ can be called the structure; each one determines the other.) Then a formula F is **true in** α if $\hat{F}(\alpha) = 1$. If every formula in a set Φ of formulas is true in a structure α, then α is a **model** of Φ. If F is true in every model of Φ, then we say that F is a **logical consequence** of Φ, or that Φ **entails** F, and we write

$$\Phi \models F.$$

A formula F is **valid**, or is a **validity**, if it is true in all structures; in that case, we write

$$\models F.$$

A proof-system $\mathcal{S}$ for propositional logic is called:

(∗) **sound**, if $\Phi \models \phi$ whenever $\Phi \vdash_{\mathcal{S}} \phi$;

(†) **complete**, if $\Phi \vdash_{\mathcal{S}} \phi$ whenever $\Phi \models \phi$.

{lem:sound}

Lemma 2.1. *Let $\mathcal{S}$ be a proof-system for propositional logic. Then $\mathcal{S}$ is sound if and only if:*

(∗) *each axiom of $\mathcal{S}$ is valid;*

(†) *$\Phi \models \phi$ whenever ϕ can be inferred from Φ by one of the rules of inference of $\mathcal{S}$.*

Proof. Suppose $\mathcal{S}$ is sound. If ϕ is an axiom of $\mathcal{S}$, then the one-term sequence ϕ is a deduction of ϕ from $\varnothing$, so $\vdash_{\mathcal{S}} \phi$ and therefore $\models \phi$. Suppose, instead, that ϕ can be inferred from Φ by one of the rules of inference of $\mathcal{S}$. Then Φ is a finite set $\{\psi_0, \dots, \psi_n\}$, so the sequence

$$\psi_0, \dots, \psi_n, \phi$$

is a deduction of ϕ from Φ in $\mathcal{S}$. Hence $\Phi \vdash_{\mathcal{S}} \phi$, and therefore $\Phi \models \phi$.

The converse is proved by induction on the lengths of deductions. Suppose that each axiom of $\mathcal{S}$ is valid, and $\Phi \models \phi$ whenever ϕ can be inferred from Φ by one of the rules of inference of $\mathcal{S}$. As an inductive hypothesis, suppose $\Phi \models \phi$ whenever ϕ has a deduction in $\mathcal{S}$ from Φ of length less than $n+1$. Now say the sequence

$$\psi_0, \dots, \psi_{n-1}, \phi$$

of length $n+1$ is a deduction in $\mathcal{S}$ from Φ. If $\phi \in \Phi$, then $\Phi \models \phi$ trivially. If ϕ is an axiom of $\mathcal{S}$, then $\models \phi$ by assumption, so $\Phi \models \phi$. The remaining possibility is that ϕ can be inferred from some subset Γ of $\{\psi_k : k < n\}$ by a rule of inference of $\mathcal{S}$. Then $\Gamma \models \phi$ by assumption. Also, $\Phi \models \psi_k$ for each ψ_k in Γ by inductive hypothesis, since each ψ_k has a proof from Φ of length $k+1$, namely

$$\psi_0, \dots, \psi_k.$$

Hence every model of Φ is a model of Γ, and so ϕ is true in this model; that is, $\Phi \models \phi$. □

Let us also note that if a proof-system is complete, then so is every proof-system obtained by addition of new axioms or rules of inference.

In the only proof-system for first-order logic that we shall consider,

(∗) the axioms are just the tautologies;

(†) the only rule of inference is *modus ponens*, that is, G can be inferred from $\{F, (F \to G)\}$.

If, in this system, F is deducible from the set Φ of formulas, then we can just write

$$\Phi \vdash F$$

(since we shall consider no other proof-systems for propositional logic). We have proved (in class) that this system is sound and complete.

3 First-order logic

{sect:1st}

The foregoing notions in propositional logic generalize to first-order logic. For us, the alphabet for a first-order logic will consist of:

(∗) the symbols in a signature $\mathcal{L}$ for the logic;

(†) individual variables v_k, where $k \in \omega$;

(‡) the Boolean connectives $\neg$ and $\to$;

(§) the quantifier $\exists$;

(¶) the brackets (and).

The set of formulas of the resulting logic can be denoted

$$\mathrm{Fm}_{\mathcal{L}}\,.$$

Certain formulas are *sentences*; the set of them is

$$\mathrm{Sn}_{\mathcal{L}}\,.$$

We do not have proof by induction on this set, since sentences can be constructed from formulas that are not sentences. However, we can still define proof-systems for $\mathrm{Sn}_{\mathcal{L}}$. (Alternatively, we could define a proof-system for $\mathrm{Fm}_{\mathcal{L}}$.)

There are **$\mathcal{L}$-structures** $\mathfrak{A}$, and then for each sentence σ of $\mathcal{L}$, there is an element $\sigma^{\mathfrak{A}}$ of 2. Then σ is **true in** $\mathfrak{A}$ if $\sigma^{\mathfrak{A}} = 1$. The notions of **model**, **entailment**, **validity**, **soundness** and **completeness** can now be defined as for propositional logic. Hence we have Lemma 2.1 for $\mathrm{Sn}_{\mathcal{L}}$ in addition to propositional logic.

To *prove* that a certain proof-system for $\mathrm{Sn}_{\mathcal{L}}$ is complete, we shall use the method first expounded by Leon Henkin, in [1]. (Henkin's proof was a part of his doctoral thesis; see [2]. We have already used Henkin's method to prove Compactness.) The particular treatment in these notes owes something to Shoenfield's in [3]. I introduce the notions of *tautological* and *deductive* completeness merely to make our ultimate proof-system seem natural.

If F is an n-ary formula $F(P_0, \dots, P_{n-1})$ of propositional logic, and $\sigma_k \in \mathrm{Sn}_{\mathcal{L}}$, then by substitution we can form the sentence

$$F(\sigma_0, \dots, \sigma_{n-1})$$

of $\mathcal{L}$. If F is a tautology, then $F(\sigma_0, \dots, \sigma_{n-1})$ can be called a **tautology** of $\mathrm{Sn}_{\mathcal{L}}$.

{lem:validities}

Lemma 3.1. *Tautologies of* $\mathrm{Sn}_{\mathcal{L}}$ *are validities.*

Proof. We can prove by induction on propositional formulas F that, if F is $F(P_0, \dots, P_{n-1})$, then for all sentences σ_k of $\mathrm{Sn}_{\mathcal{L}}$, and all $\mathcal{L}$-structure $\mathfrak{A}$,

$$F(\sigma_0, \dots, \sigma_{n-1})^{\mathfrak{A}} = \hat{F}({\sigma_0}^{\mathfrak{A}}, \dots, {\sigma_{n-1}}^{\mathfrak{A}}).$$

(Details are an **exercise**.) The claim follows immediately from this. □ {ex2}

4 Tautological completeness

Suppose $\mathcal{S}$ is a proof-system for $\mathrm{Sn}_{\mathcal{L}}$ such that, if F_0, …, F_k are n-ary propositional formulas, and

{eqn:prop-ent}
$$\{F_0, \dots, F_{k-1}\} \models F_k, \tag{1}$$

and $\sigma_0, \dots, \sigma_{n-1} \in \mathrm{Sn}_{\mathcal{L}}$, then

{eqn:1-ent}
$$\{F_0(\sigma_0, \dots, \sigma_{n-1}), \dots, F_{k-1}(\sigma_0, \dots, \sigma_{n-1})\} \vdash_{\mathcal{S}} F_k(\sigma_0, \dots, \sigma_{n-1}); \tag{2}$$

let us say then that $\mathcal{S}$ is **tautologically complete**.

{lem:1}

Lemma 4.1. *Let $\mathcal{S}$ be a proof-system for $\mathrm{Sn}_{\mathcal{L}}$. Then $\mathcal{S}$ is tautologically complete if and only if:*

(∗) $\vdash_{\mathcal{S}} \sigma$ *for all tautologies σ of $\mathrm{Sn}_{\mathcal{L}}$, and*

(†) $\{\sigma, \sigma \to \tau\} \vdash_{\mathcal{S}} \tau$ *for all σ and τ in $\mathrm{Sn}_{\mathcal{L}}$.*

Proof. If $\mathcal{S}$ is tautologically complete, then immediately all tautologies are theorems; the other condition follows since $\{P_0, P_0 \to P_1\} \models P_1$.

To prove the converse, we can use our complete proof-system for propositional logic: Suppose we have (1) above. Then F_k has a a formal proof from $\{F_0, \dots, F_{k-1}\}$. Say this proof is

$$G_0, \dots, G_m.$$

Then G_m is F_k. We proceed by induction on m. There are three possibilities:

(∗) If $F_k \in \{F_0, \dots, F_{k-1}\}$, then trivially (2) follows.

(†) If F_k is a tautology, then $\vdash_{\mathcal{S}} F_k(\vec{\sigma})$ by assumption, so (2).

(‡) If G_j is $(G_i \to F_k)$ for some i and j in m, then, by inductive hypothesis, we have

$$\{F_0(\vec{\sigma}), \dots, F_{k-1}(\vec{\sigma})\} \vdash_{\mathcal{S}} G_i(\vec{\sigma});$$
$$\{F_0(\vec{\sigma}), \dots, F_{k-1}(\vec{\sigma})\} \vdash_{\mathcal{S}} G_j(\vec{\sigma});$$

hence (2) by assumption (and Lemma 1.1).

In all cases then, (2) follows. □

It should be clear that a complete proof-system is tautologically complete. The converse fails:

Example 4.2. The proof-system in which all tautologies are axioms and *modus ponens* is the only rule of inference is not complete, since it cannot be used to prove the validity $\exists x\ x = x$. Indeed, the theorems of this proof-system are just the tautologies (as one can show); but $\exists x\ x = x$ is not a tautology.

Let $\bot$ be the negation of a tautology, say

$$\neg(\exists x\ x = x \to \exists x\ x = x).$$

Henceforth, let $\Sigma \subseteq \mathrm{Sn}_{\mathcal{L}}$ and $\sigma \in \mathrm{Sn}_{\mathcal{L}}$. {lem:2}

Lemma 4.3. *In a tautologically complete proof-system $\mathcal{S}$, the following are equivalent:*

(∗) $\Sigma \vdash \neg\sigma$ *for some* σ *in* Σ*;*

(†) $\Sigma \vdash \sigma$ *and* $\Sigma \vdash \neg\sigma$ *for some* σ *in* $\mathrm{Sn}_{\mathcal{L}}$*;*

(‡) $\Sigma \vdash \sigma$ *for every* σ *in* $\mathrm{Sn}_{\mathcal{L}}$*;*

(§) $\Sigma \vdash \bot$.

Proof. **Exercise.** (There is a corresponding lemma for propositional logic.) □ {ex3}

If $\Sigma \vdash_{\mathcal{S}} \bot$, then Σ is **inconsistent** in $\mathcal{S}$; otherwise, it is **consistent**.

Lemma 4.4. *In a complete proof-system, every consistent subset of* $\mathrm{Sn}_{\mathcal{L}}$ *has a model.*

Proof. If $\mathcal{S}$ is complete, but Σ has no model, then $\Sigma \models \bot$, so $\Sigma \vdash_{\mathcal{S}} \bot$ by completeness, so Σ is inconsistent. □

The converse of the lemma may fail, even if the proof-system is required to be tautologically complete:

Example 4.5. Let the axioms of a proof-system $\mathcal{S}$ be the tautologies, and let the rules of inference be *modus ponens*, along with the rule that $\bot$ can be inferred from every finite set that has no model. (Note however that this is not a *syntactical* rule: it is not based directly on the form of sentences.) By the Compactness Theorem of first-order logic, *every* set with no model is inconsistent in this theory; therefore all consistent sets have models. However, the validity $\exists x\ x = x$ is not a theorem of $\mathcal{S}$. (**Exercise:** show this.) {ex4}

5 Deductive completeness

Let a proof-system $\mathcal{S}$ be called **deductively complete** if $\Sigma \vdash_{\mathcal{S}} (\sigma \to \tau)$ whenever $\Sigma \cup \{\sigma\} \vdash_{\mathcal{S}} \tau$. {lem:4}

Lemma 5.1. *A tautologically and deductively complete proof-system in which every consistent set has a model is complete.*

Proof. Suppose $\mathcal{S}$ is such a system, and $\Sigma \cup \{\neg\sigma\}$ is inconsistent in $\mathcal{S}$. Then $\Sigma \cup \{\neg\sigma\} \vdash_{\mathcal{S}} \sigma$ by Lemma 4.3, so $\Sigma \vdash_{\mathcal{S}} (\neg\sigma \to \sigma)$ by deductive completeness. But $(\neg\sigma \to \sigma) \to \sigma$ is a tautology, so $\Sigma \vdash_{\mathcal{S}} \sigma$ by tautological completeness.

Therefore, if $\Sigma \nvdash_{\mathcal{S}} \sigma$, then $\Sigma \cup \{\neg\sigma\}$ is consistent, so it has a model by assumption; this shows $\Sigma \not\models \sigma$. □

{lem:5}

Lemma 5.2. *A tautologically complete proof-system whose only rule of inference is* modus ponens *is deductively complete.*

{ex5} *Proof.* **Exercise.** (See the Deduction Theorem of propositional logic.) □

Lemma 5.3. *Suppose $\Sigma \subseteq \mathrm{Sn}_{\mathcal{L}}$ and Σ is consistent in a tautologically and deductively complete proof-system. The following are equivalent:*

(∗) *If $\Sigma \subseteq \Gamma \subseteq \mathrm{Sn}_{\mathcal{L}}$ and Γ is consistent, then $\Gamma = \Sigma$.*

(†) $\neg\sigma \in \Sigma \iff \sigma \notin \Sigma$ *for all σ in $\mathrm{Sn}_{\mathcal{L}}$.*

{ex6} *Proof.* **Exercise.** □

A set Σ meeting one of the conditions in the lemma can be called **maximally** consistent.

6 Completeness

By Lemma 4.1, we know of one tautologically complete proof-system, namely, the system whose axioms are the tautologies, and whose rule of inference is *modus ponens.* Let $\mathcal{S}$ be this system. Then $\mathcal{S}$ is deductively complete, by Lemma 5.2, and is sound, by Lemmas 2.1 and 3.1. Moreover, soundness and deductive completeness are preserved if we add new valid axioms to $\mathcal{S}$. Now we shall see which valid axioms we can add in order to ensure that every consistent set has a model; then we shall have a complete system by Lemma 5.1.

We follow the proof of the Compactness Theorem, replacing 'finitely satisfiable' with 'consistent'. We assume that $\mathcal{L}$ is countable. Suppose Σ is a consistent subset of $\mathrm{Sn}_{\mathcal{L}}$. We introduce an infinite set C of new constants and enumerate $\mathrm{Sn}_{\mathcal{L}\cup C}$ as $\{\sigma_n : n \in \omega\}$. We construct a chain

$$\Sigma = \Sigma_0 \subseteq \Sigma_1 \subseteq \Sigma_2 \subseteq \cdots$$

where

$$\Sigma_{2n+1} = \begin{cases} \Sigma_{2n} \cup \{\sigma_n\}, & \text{if this is consistent;} \\ \Sigma_{2n}, & \text{otherwise.} \end{cases}$$

If σ_n is $\exists x\ \phi$, and this is in Σ_{2n+1}, then we want to define Σ_{2n+2} as

$$\Sigma_{2n+1} \cup \{\phi^x_c\},$$

where c is a variable not used in Σ_{2n+1}. But we need to know that this set is consistent. For this we assume, as axioms of $\mathcal{S}$, the sentences

{eqn:axiom-E}

$$(\phi^x_c \to \chi) \to \exists x\ \phi \to \chi, \tag{3}$$

where c is a variable not appearing in χ. Note that these axioms are valid. We now have:

Lemma 6.1. *If Γ is consistent and contains $\exists x\ \phi$, and c does not appear in Γ, then $\Gamma \cup \{\phi^x_c\}$ is consistent.*

Proof. Suppose it's not. Then

$$\{\psi_0, \dots, \psi_{k-1}\} \cup \{\phi^x_c\} \vdash_{\mathcal{S}} \bot$$

for some ψ_i in Γ. By deductive completeness,

$$\vdash_{\mathcal{S}} \phi^x_c \to \psi_0 \to \dots \to \psi_{k-1} \to \bot, \tag{4}$$ {eqn:to-bot}

where the notational convention is that a terminal string $\chi_0 \to \chi_1 \to \chi_2$ stands for the formula $(\chi_0 \to (\chi_1 \to \chi_2))$. We can re-write (4) as

$$\vdash_{\mathcal{S}} \phi^x_c \to \chi, \tag{5}$$ {eqn:to-bot-again}

where χ is $\psi_0 \to \dots \to \psi_{k-1} \to \bot$. Then from (3) we have

$$\vdash_{\mathcal{S}} \exists x\ \phi \to \chi$$

by *modus ponens*; that is,

$$\vdash_{\mathcal{S}} \exists x\ \phi \to \psi_0 \to \dots \to \psi_{k-1} \to \bot.$$

Then $k+1$ applications of *modus ponens* show

$$\Gamma \vdash_{\mathcal{S}} \bot,$$

which contradicts the assumption that Γ is consistent. □

So now, given a consistent subset Σ of $\mathrm{Sn}_{\mathcal{L}}$, we can construct a consistent subset Σ^* of $\mathrm{Sn}_{\mathcal{L}\cup C}$ such that

(∗) $\Sigma \subseteq \Sigma^*$;

(†) Σ^* is maximally consistent;

(‡) if $(\exists x\ \phi) \in \Sigma$, then $\phi^x_c \in \Sigma$ for some c in C, that is, Σ^* **has witnesses**.

As in the proof of Compactness, we want to use Σ^* to define a model $\mathfrak{A}$ of itself.

For the sake of defining the universe of $\mathfrak{A}$, we assume now that $\mathcal{S}$ has the axioms

$$c = c, \tag{6}$$ {eqn:equality}

$$c = c' \to d = d' \to c = d \to c' = d', \tag{7}$$ {eqn:more-equality}

where c, c', d and d' range over C. Let E be the relation

$$\{(c, d) \in C^2 : (c = d) \in \Sigma^*\}.$$

We can now show:

Lemma 6.2. *The relation E is an equivalence-relation.*

Proof. We first show

$$\vdash_{\mathcal{S}} c = c, \tag{8}$$ {eqn:=1}

$$\vdash_{\mathcal{S}} c = d \to d = c, \tag{9}$$ {eqn:=2}

$$\vdash_{\mathcal{S}} c = d \to d = e \to c = e \tag{10}$$ {eqn:=3}

for all constants c, d and e in C.

Now, we have (8) trivially by (6). An instance of (7) is

$$c = d \to c = c \to c = c \to d = c;$$

then (9) follows by tautological completeness. Another instance of (7) is

$$c = c \to d = e \to c = d \to c = e;$$

then (10) follows by tautological completeness.

By its maximal consistency then, Σ^* contains $c = c$; and if Σ^* contains $c = d$ and $d = e$, then it contains $d = c$ and $c = e$. □

We define A to be C/E. We now define $R^{\mathfrak{A}}$ (for each n-ary predicate R in $\mathcal{L}$) as the set

$$\{([c_0], \cdots, [c_{n-1}]) \in A^n : (Rc_0 \cdots c_{n-1}) \in \Sigma^*\}.$$

Then we have

$$(Rc_0 \cdots c_{n-1}) \in \Sigma^* \implies ([c_0], \cdots, [c_{n-1}]) \in R^{\mathfrak{A}},$$

but perhaps not the converse. Possibly then both $Rc_0 \cdots c_{n-1}$ and $\neg Rc'_0 \cdots c'_{n-1}$ are in Σ^*, although $(c_k = c'_k) \in \Sigma^*$ in each case. To prevent this, as as axioms of $\mathcal{S}$ we assume

{eqn:R} $$c_0 = c'_0 \to \cdots \to c_{n-1} = c'_{n-1} \to Rc_0 \cdots c_{n-1} \to Rc'_0 \cdots c'_{n-1}. \tag{11}$$

We now have:

Lemma 6.3. $([c_0], \cdots, [c_{n-1}]) \in R^{\mathfrak{A}} \iff (Rc_0 \cdots c_{n-1}) \in \Sigma^*$.

{ex7} *Proof.* **Exercise.** □

Finally, suppose f is an n-ary function-symbol (where possibly $n = 0$, in which case f is a constant.) We want to be able to define $f^{\mathfrak{A}}$. (If $c \in C$, then $c^{\mathfrak{A}} = [c]$; but there might be constants of $\mathcal{L}$ as well.) To define $f^{\mathfrak{A}}$, we first need some lemmas, which are based on another axiom:

{eqn:t} $$\phi^x_t \to \exists x\, \phi, \tag{12}$$

where $\mathrm{fv}(\phi) \subseteq \{x\}$ and t is a term with no variables. Let us assume that this is an axiom of $\mathcal{S}$. Then we have:

Lemma 6.4 (Substitution). *If* $\mathrm{fv}(\phi) \subseteq \{x\}$, *and the constant* c *does not appear in* ϕ, *then*

$$\vdash_{\mathcal{S}} \phi^x_c \to \phi^x_t$$

for all constant terms t.

Proof. We have

$$\begin{aligned}
&\vdash_{\mathcal{S}} \neg\phi^x_t \to \exists x\, \neg\phi, && \text{[by (12)]}\\
&\vdash_{\mathcal{S}} \neg\exists x\, \neg\phi \to \phi^x_t, && \text{[by tautological completeness]}\\
&\vdash_{\mathcal{S}} (\neg\phi^x_c \to \bot) \to \exists x\, \neg\phi \to \bot, && \text{[by (3)]}\\
&\vdash_{\mathcal{S}} \phi^x_c \to \neg\exists x\, \neg\phi, && \text{[by tautological completeness]}
\end{aligned}$$

and hence $\vdash_{\mathcal{S}} \phi^x_c \to \phi^x_t$ by *modus ponens.* □

Lemma 6.5. *$\vdash_{\mathcal{S}} t = t$ for all terms t.*

Proof. We have

$$\begin{aligned}
&\vdash_{\mathcal{S}} c = c, && \text{[by (6)]}\\
&\vdash_{\mathcal{S}} c = c \to t = t, && \text{[by the Substitution Lemma]}
\end{aligned}$$

and hence $\vdash_{\mathcal{S}} t = t$ by *modus ponens.* □

Lemma 6.6. *$\vdash_{\mathcal{S}} \exists x\, fc_0 \cdots c_{n-1} = x$.*

Proof. We have

$$\begin{aligned}
&\vdash_{\mathcal{S}} fc_0 \cdots c_{n-1} = fc_0 \cdots c_{n-1}, && \text{[by the last lemma]}\\
&\vdash_{\mathcal{S}} fc_0 \cdots c_{n-1} = fc_0 \cdots c_{n-1} \to \exists x\, fc_0 \cdots c_{n-1} = x, && \text{[by (12)]}
\end{aligned}$$

hence $\vdash_{\mathcal{S}} \exists x\, fc_0 \cdots c_{n-1} = x$ by *modus ponens.* □

Finally, we assume as axioms of $\mathcal{S}$ the sentences

$$c_0 = c'_0 \to \cdots \to c_{n-1} = c'_{n-1} \to fc_0 \cdots c_{n-1} = fc'_0 \cdots c'_{n-1}. \qquad (13)$$ {eqn:f}

This enables us to define $f^{\mathfrak{A}}$:

Lemma 6.7. *For each n-ary function-symbol f, there is an n-ary operation $f^{\mathfrak{A}}$ on A given by*

$$f^{\mathfrak{A}}([c_0], \dots, [c_{n-1}]) = [d] \iff (fc_0 \cdots c_{n-1} = d) \in \Sigma^*. \qquad (14)$$ {eqn:function}

Proof. Since Σ^* is maximally consistent, we now have

$$\exists x\, fc_0 \cdots c_{n-1} = x \in \Sigma^*.$$

Since Σ^* has witnesses, we have

$$fc_0 \cdots c_{n-1} = d \in \Sigma^*$$

for some constant d. This gives us a value for $f^{\mathfrak{A}}([c_0], \cdots, [c_{n-1}])$; we have to show that this value is unique. For this, it is enough to show

$$\begin{aligned}
\vdash_{\mathcal{S}} c_0 = c'_0 \to \cdots \to c_{n-1} = c'_{n-1} \to&\\
&d = d' \to fc_0 \cdots c_{n-1} = d \to fc'_0 \cdots c'_{n-1} = d'
\end{aligned}$$

for all c_k and c'_k and d and d' in C. By (13) and tautological completeness, it is enough to show

$$\vdash_{\mathcal{S}} fc_0 \cdots c_{n-1} = fc'_0 \cdots c'_{n-1} \to d = d' \to fc_0 \cdots c_{n-1} = d \to fc'_0 \cdots c'_{n-1} = d'.$$

In the axiom (7), we may assume that c is not one of the variables c', d or d'. Then by the Substitution Lemma, we have

$$\vdash_{\mathcal{S}} fc_0 \cdots c_{n-1} = c' \to d = d' \to fc_0 \cdots c_{n-1} = d \to c' = d'.$$

We may also assume that c' is not one of the variables c_k, d or d'. Applying the Substitution Lemma again gives what we want. □

The structure $\mathfrak{A}$ is now determined and is a model of Σ, by the proof of the Compactness Theorem. In sum, what we have shown is:

{thm:completeness}

Theorem 6.8 (Completeness for first-order logic). *That proof-system for* $\mathrm{Sn}_{\mathcal{L}}$ *is complete whose only rule of inference is* modus ponens*, and whose axioms are the following:*

- (∗) *the tautologies;*
- (†) $(\phi^x_c \to \chi) \to \exists x\ \phi \to \chi$, *where c does not appear in χ;*
- (‡) $c = c$;
- (§) $c = c' \to d = d' \to c = d \to c' = d'$;
- (¶) $c_0 = c'_0 \to \ldots c_{n-1} = c'_{n-1} \to Rc_0 \cdots c_{n-1} \to Rc'_0 \cdots c'_{n-1}$;
- (‖) $\phi^x_t \to \exists x\ \phi$;
- (∗∗) $c_0 = c'_0 \to \cdots \to c_{n-1} = c'_{n-1} \to fc_0 \cdots c_{n-1} = fc'_0 \cdots c'_{n-1}$.

Here the notation is as follows:

- x *is a variable;*
- ϕ *is a formula such that* $\mathrm{fv}(\phi) \subseteq \{x\}$*;*
- χ *is a sentence;*
- t *is a constant term;*
- c, c', c_k, c'_k, d *and* d' *are constants;*
- $n \in \omega$*;*
- R *is an n-ary predicate if* $n > 0$*; and*
- f *is an n-ary function-symbol (or a constant, if* $n = 0$*).*

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