ANCIENT GREEK MATHEMATICS

DAVID PIERCE

Contents

1. Introduction

Some time in the 3rd century B.C.E., Apollonius of Perga wrote eight books on **conic sections.** We have the first four books [2, 3] in the original Greek; the next three books survive in Arabic translation [1]; the eighth book is lost. As Apollonius tells us in an introductory letter, his first four books are part of an elementary course on the conic sections.

Before Apollonius, around 300 B.C.E., Euclid published the thirteen books of the *Elements* [6, 8, 9], a work of mathematics of which some of parts could well be used as a textbook today. The *Elements* provide a good example of mathematical exposition and of what it means to prove something.

In 2008, getting ready to teach a course on the conic sections[1], I wrote some notes on ancient mathematics. Using those notes, I have prepared the present notes, for use in a course called 'History of Mathematical Concepts I' at METU—a course in which participants will read Euclid and Apollonius.

In the latter sections of these notes, I look at some general features of ancient mathematics as I understand it. Meanwhile, in § 3, I jump forward in history to Descartes, to see the sorts of improvements that he thought he was making to mathematical practice of mathematicians like Euclid and Apollonius.

Because I shall occasionally refer to some Greek words, I review the Greek alphabet in Table 1. (I have heard a rumor that students can improve their mathematics simply by learning this alphabet, assuming they didn't grow up knowing it.)

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[1]At the Nesin Mathematics Village, Şirince, Selçuk, İzmir, Turkey.

A	α	alpha	H	η	ēta	N	ν	nu	T	τ	tau
B	β	beta	Θ	θ	theta	Ξ	ξ	xi	Υ	υ	upsilon
Γ	γ	gamma	I	ι	iota	O	o	omicron	Φ	ϕ	phi
Δ	δ	delta	K	κ	kappa	Π	π	pi	X	χ	chi
E	ϵ	epsilon	Λ	λ	lambda	P	ρ	rho	Ψ	ψ	psi
Z	ζ	zeta	M	μ	mu	Σ	σ,ς	sigma	Ω	ω	ōmega

TABLE 1. The first letter or two of the (Latin) name for a Greek letter provides a transliteration for that letter. However, upsilon is also transliterated by *y*. The diphthong αι often comes into English (*via* Latin) as *ae*, while οι may come as *oe*. The second form of the small sigma is used at the ends of words. In texts, the rough-breathing mark (ʽ) over an initial vowel (or ρ) is transcribed as a preceeding (or following) h (as in ὁ ῥόμβος *ho rhombos* 'the rhombus'). The smooth-breathing mark (ʼ) and the three tonal accents (ά, ᾶ, ὰ) can be ignored. Especially in the dative case (the Turkish -e hali), some long vowels may be given the iota subscript (ᾳ, ῃ, ῳ), representing what was once a following iota (αι, ηι, ωι).

2. Why read the Ancients?

As an undergraduate, I attended a college[2] where Euclid and Apollonius were used as textbooks. They were so used, I think, not because they were considered to be the *best* textbooks, but because they *had been* textbooks for countless generations of mathematicians: therefore (the idea was), one might gain some understanding of humanity and oneself by reading these books. (The same is true for Homer, Aeschylus, Plato, and the other great books read at the college.)

Now, having become a professional mathematician, I ask what Euclid and Apollonius have to offer the mathematician of today. It is in pursuit of an answer to this question that I prepare these notes—which therefore are part of an ongoing project.

I prepare these notes also for the sake of honesty about what students are asked to learn. The curves called **conic sections** are a standard part of an elementary course of mathematics. The origin of such curves is in the name: they are obtained by slicing a cone. Apollonius treated the curves in this way. But in math courses today, the conic sections are usually given as the curves defined by certain equations, such as

$$ay = x^2$$

or

$$\frac{x^2}{a^2} \pm \frac{y^2}{b^2} = 1.$$

Or perhaps the curves are given in terms of foci and directrices. A textbook may *assert* that the curves so defined can indeed by obtained as sections of cones; but it is rare that this assertion is justified.

One calculus textbook[3] writes:

[2]St. John's College, with campuses in Annapolis, Maryland, and Santa Fe, New Mexico, USA.

[3]James Stewart, *Calculus,* fifth edition, p. 720. This text is currently in use at METU.

> In this section we give geometric definitions of parabolas, ellipses, and hyperbolas and derive their standard equations. They are called **conic sections,** or **conics,** because they result from intersecting a cone with a plane as shown in Figure 1.

(I omit the author's figure.) The conic sections result from intersecting a cone with a plane: this can be understood as a *definition* of the conic sections. Let us call it Definition I. More precisely, this definition distinguishes three kinds of conic sections, depending on the angle of the plane with respect to the cone. One kind of conic section is called the *parabola,* and the text continues under the heading *Parabolas:*

> A **parabola** is the set of points in a plane that are equidistant from a fixed point F (called the **focus**) and a fixed line (called the **directrix**)... In the 16th century Galileo showed that the path of a projectile that is shot into the air at an angle to the ground is a parabola. Since then, parabolic shapes have been used in designing automobile headlights, reflecting telescopes, and suspension bridges... We obtain a particularly simple equation for a parabola if we place its vertex at the origin O...

Here then is another definition of the parabola; call it Definition II. Definitions I and II are equivalent in that they define the same objects; but the author does not clearly say so, much less prove it. I don't think he *needs* to prove the equivalence; but at least he ought to state that he is not going to prove it.

Perhaps the author expects the reader to *infer* the equivalence of Definitions I and II. But this is not his style. He is usually eager to give his readers every assistance. Note for example that he apparently does not trust readers to infer for themselves that parabolas are worth studying. Before concluding anything from his definition of parabolas, the author feels the need to tell the reader how *useful* parabolas are.

Another textbook[4] follows a similar procedure, first defining the conic sections *as such,* then defining them in terms of foci and directrices. Between the two definitions, the writer observes that the intersection of a cone and a plane will be given by a second-degree equation. This suggests that the quadratic equations to be derived presently in the book may indeed define conic sections. However, no attempt is made to prove that every curve defined by a quadratic equation can be obtained as the section of a cone. The author observes:

> After straight lines the conic sections are the simplest of plane curves. They have many properties that make them useful in applications of mathematics; that is why we include a discussion of them here. Much of this material is optional from the point of view of a calculus course, but familiarity with the properties of conics can be very important in some applications. Most of the properties of conics were discovered by the Greek geometer, Apollonius of Perga, about 200 BC. It is remarkable that he was able to obtain these properties using only the techniques of classical Euclidean geometry; today most of these properties

[4]Robert A. Adams, *Calculus: a complete course,* fourth edition, p. 476. This text was formerly used at METU.

> are expressed more conveniently using analytic geometry and specific coordinate systems.

Again, the justification offered for the study of the conic sections is their usefulness. But as for 'expressing' the properties of conic sections, which of the following expresses better what a conic section *is?*

(i) It is the intersection of a cone and a plane.
(ii) It is the intersection of the surfaces defined by the equations

$$ax + by + cz + d = 0,$$
$$(x - ez)^2 + y^2 = fz^2.$$

What the author means, I think, is that it is convenient to define certain curves 'analytically'—that is, in a coordinate system such as Descartes introduced; properties of the curves can then be obtained by further analysis. But showing that those curves are conic sections is a whole other problem, not addressed in the book.

By the way, despite what the last quotation suggests, I am not sure that obtaining nice results with limited mathematical tools is remarkable in itself. The tools of an artisan depend on what is available in the physical environment; but the tools of a mathematician depend only on imagination. A mathematician without the imagination to come up with the best tool for the job would seem to be an unremarkable mathematician.

The first chapter of Hilbert and Cohn-Vossen's *Geometry and the Imagination* [10] contains a beautiful account of how various properties of the conic sections arise from consideration of the cones from which the sections are obtained. However, the cones considered by the authors are all *right* cones. Apollonius does not make this restriction. Hilbert and Cohn-Vossen give an etymology for the names of the ellipse, the hyperbola, and the parabola: it involves eccentricity. The etymology is plausible, but it appears to be literally incorrect, as a reading of Book I of Apollonius would show.

Mathematics reveals underlying correspondences between seemingly dissimilar things. Sometimes we treat these correspondences as identities. This can be a mistake. There is a correspondence between conic sections and quadratic equations. But are the sections *really* the equations? One cannot answer the question without considering *conic sections as such,* as Apollonius considered them.

3. Synthesis and analysis

It may be said that, in reading Euclid and Apollonius, we are going to do **pre-Cartesian** mathematics: mathematics as done before (well before) the time of René Descartes (1596–1650).

The geometry pioneered by René Descartes is called **analytic geometry;** by contrast, the geometry of ancient mathematicians like Euclid and Apollonius is sometimes called **synthetic geometry.** But what does this *mean?* The word *synthetic* comes from the Greek **συνθετικός,** meaning *skilled in putting together* or *constructive.* This Greek adjective derives from the verb **συντίθημι** *put together, construct* (from **συν** *together* and **τίθημι** *put*). The word *analytic* is the English form of **ἀναλυτικός**, which derives from the verb **ἀναλύω** *undo, set free, dissolve* (from **ἀνα** *up,* **λύω** *loose*). Although we refer to ancient geometry as synthetic, the Ancients evidently recognize both analytic and synthetic methods. Around 320 C.E., Pappus of Alexandria writes [14, p. 597]:

> Now **analysis** (ἀνάλυσις) is a method of taking that which is sought as though it were admitted and passing from it through its consequences in order to something which is admitted as a result of synthesis; for in analysis we suppose that which is sought to be already done, and we inquire what it is from which this comes about, and again what is the antecedent cause of the latter, and so on until, by retracing our steps, we light upon something already known or ranking as a first principle; and such a method we call analysis, as being a *reverse solution* (ἀνάπαλιν λύσις).
>
> But in **synthesis** (συνθέσις), proceeding in the opposite way, we suppose to be already done that which was last reached in the analysis, and arranging in their natural order as consequents what were formerly antecedents and linking them one with another, we finally arrive at the construction of what was sought; and this we call synthesis.
>
> Now analysis is of two kinds, one, whose object is to seek the truth, being called **theoretical** (θεωρητικός), and the other, whose object is to find something set for finding, being called **problematical** (προβληματικός).

This passage is not very useful without examples: I shall propose one presently. Meanwhile, I that Pappus elsewhere [14, pp. 564–567] says more about the distinction between theorems and problems:

> Those who favor a more technical terminology in geometrical research use **problem** (πρόβλημα) to mean a [proposition[5]] in which it is proposed to do or construct [something]; and **theorem** (θεώρημα), a [proposition] in which the consequences and necessary implications of certain hypotheses are investigated; but among the ancients some described them all as problems, some as theorems.

What really distinguishes Cartesian geometry from what came before is perhaps suggested by the first sentence of Descartes's *Geometry* [4, p. 2]:

> Any problem in geometry can easily be reduced to such terms that a knowledge of the lengths of certain straight lines is sufficient for its construction.

From a straight line, Descartes abstracts something called *length.* A length is something that we might today call a positive real number.

Descartes takes the edifice of geometry that has been built up or 'synthesized' over the centuries, and reduces or 'analyzes' its study into the manipulation of numbers. To be more precise, he 'takes that which is sought as though it were admitted' in the following way. In Figure 1, straight lines BE, DR, and FS are given in position (meaning their endpoints themselves are not fixed); and the sizes of angles ABC, ADC, and CFE are given. It is required to find the point C so that the rectangle with sides BC and CD has a given ratio to the square on CF. (This is a simplified version of the problem that Descartes takes up in the *Geometry.*)

[5]Ivor Thomas [14, p. 567] uses *inquiry* here in his translation; but there is *no* word in the Greek original corresponding to this or to *proposition.*

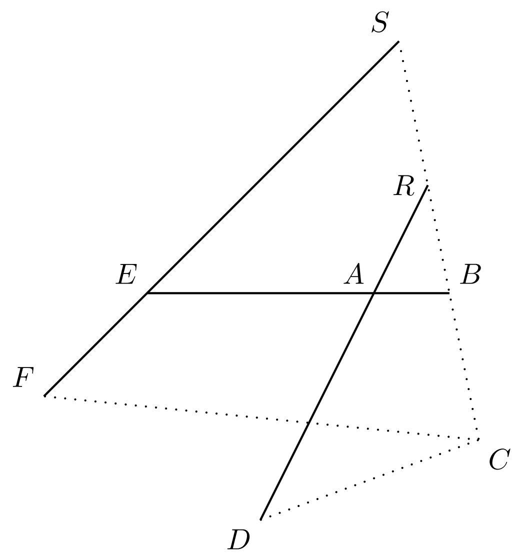

Figure 1

In his analytic approach, Descartes assumes that C has already been found, as in the figure. We denote AB by x, and BC by y. The ratio $AB : BR$ is given; call it $z : b$. Then

$$RB = \frac{bx}{z}, \qquad CR = y + \frac{bx}{z} = \frac{zy + bx}{z}.$$

But $CR : CD$ is given; call it $z : c$. Then

$$CD = \frac{czy + bcx}{z^2}.$$

Also AE is given; call it k. And let $BE : BS = z : d$. Then

$$BE = k + x, \quad BS = \frac{dk + dx}{z}, \quad CS = \frac{zy + dk + dx}{z}.$$

Finally, if $CS : CF = z : e$, then

$$CF = \frac{ezy + dek + dex}{z^2}.$$

So it is given that the ratio

$$y \cdot \frac{czy + bcx}{z^2} : \left(\frac{ezy + dek + dex}{z^2}\right)^2$$

is constant. This gives us a quadratic equation in the unknowns x and y.

Descartes's method does not use explicitly drawn *axes* with respect to which x and y are measured. Also, the straight lines called x and y are not required to be perpendicular: they are merely not parallel.

Through analysis, we have found an equation that determines the point C. Since the equation is quadratic, the point C lies on (a curve that turns out to be) a conic section.

When there are more straight lines in the problem, then the resulting equation may have a higher degree.

We do not get any sense here for what the curve of C *looks like.* We might get some sense by analyzing the equation for C. Apollonius will give us a sense for what conic sections look like by showing *how they are related to the cones that they come from.*

4. Theorems and problems

The text of Apollonius as we have it consists almost entirely of theorems and problems (in the sense of the last section). There are some introductory remarks, some definitions, but nothing else. The theorems and problems can be analyzed in a way described by Proclus,[6] in the fifth century C.E., in his commentaries on Euclid [13, p. 159]:

> Every problem and every theorem that is furnished with all its parts should contain the following elements: an *enunciation* (πρότασις), an *exposition* (ἔκθεσις), a *specification* (διορισμός), a *construction* (κατασκευή), a *proof* (ἀπόδειξις), and a *conclusion* (συμπέρασμα). Of these, the enunciation states what is given and what is being sought from it, for a perfect enunciation consists of both these parts. The exposition takes separately what is given and prepares it in advance for use in the investigation. The specification takes separately the thing that is sought and makes clear precisely what it is. The construction adds what is lacking in the given for finding what is sought. The proof draws the proposed inference by reasoning scientifically from the propositions that have been admitted. The conclusion reverts to the enunciation, confirming what has been proved.
>
> So many are the parts of a problem or a theorem. The most essential ones, and those which are always present, are enunciation, proof, and conclusion.

Alternative translations are: for ἔκθεσις, *setting out,* and for διορισμός, *definition of goal* [12, p. 10].

For an illustration, we may analyze Proposition 1 of Book I of Euclid's *Elements* (in Fitzpatrick's translation [9]). The proposition is a *problem:*

Enunciation. *To construct an equilateral triangle on a given finite straight-line.*

Exposition. Let AB be the given finite straight-line.

Specification. So it is required to construct an equilateral triangle on the straight-line AB.

Construction. Let the circle BCD with center A and radius AB have been drawn, and again let the circle ACE with center B and radius BA have been drawn. And let the straight-lines CA and CB have been joined from the point C, where the circles cut one another, to the points A and B (respectively).

[6] Proclus was born in Byzantium (that is, Constantinople, now İstanbul), but his parents were from Lycia (Likya), and he was educated first in Xanthus. He moved to Alexandria, then Athens, to study philosophy [13, p. xxxix].

Proof. And since the point A is the center of the circle CDB, AC is equal to AB. Again, since the point B is the center of the circle CAE, BC is equal to BA. But CA was also shown (to be) equal to AB. Thus, CA and CB are each equal to AB. But things equal to the same thing are also equal to one another. Thus, CA is also equal to CB. Thus, the three (straight- lines) CA, AB, and BC are equal to one another. □

Conclusion. Thus, the triangle ABC is equilateral, and has been constructed on the given finite straight-line AB. (Which is) the very thing it was required to do.

5. Conversational implicature

One apparent difference between the ancient and modern approaches to mathematics may result from a modern habit that is exemplified in a Russian textbook of the Soviet period [5, pp. 9 f.]:

> The student of mathematics must at all times have a clear-cut understanding of all fundamental mathematical concepts. . . The student will also recall the signs of weak inequalities: $\leqslant$ (less than or equal to) and $\geqslant$ (greater than or equal to). The student usually finds no difficulty when using them in formal transformations, but examinations have shown that many students do not fully comprehend their meaning.
>
> To illustrate, a frequent answer to: "*Is the inequality* $2 \leqslant 3$ *true?*" is "No, since the number 2 is less than 3." Or, say, "*Is the inequality* $3 \leqslant 3$ *true?*" the answer is often "No, since 3 is equal to 3." Nevertheless, students who answer in this fashion are often found to write the result of a problem as $x \leqslant 3$. Yet their understanding of the sign $\leqslant$ between concrete numbers signifies that not a single specific number can be substituted in place of x in the inequality $x \leqslant 3$, which is to say that the sign $\leqslant$ cannot be used to relate any numbers whatsoever.

The students referred to, who will not allow that $2 \leqslant 3$, are following a habit of ordinary language, whereby the *whole* truth must be told. According to this habit, one does not say $2 \leqslant 3$, because one can make a stronger, more informative statement, namely $2 < 3$. This habit would appear to be an instance of *conversational implicature:* this is the ability of people to convey or *implicate* statements that are not logically *implied* by their words [11, ch. 1, §5, pp. 36–40]. In saying *A or B [is true],* one usually 'implicates' that one does not know *which* is true.

This habit of implicature may be reflected in the ancient understanding, according to which *one* (ἕν) is not a *number* (ἀριθμός). In Book VII of the *Elements,* Euclid somewhat obscurely defines a **unit** (μονάς) as that by virtue of which each being is called 'one'. (This English version of the definition is based on the Greek text supplied in [6, Vol. 2, p.279].) Then a **number** is defined as a *multitude* (πλῆθος) composed of units. In particular, a unit is not a number, because it is not a multitude: it is one. Euclid does not bother to state explicitly this distinction between units and numbers, but it can be inferred, for example, from his presentation of what we now call the Euclidean algorithm. Proposition VII.1 of the *Elements* involves a pair of numbers such that the algorithm, when applied to them, yields a unit (μονάς). Then this unit is *not* considered as a greatest common divisor of the numbers; the numbers do not *have* a greatest common divisor; the numbers are simply relatively prime. If the numbers are *not* relatively prime, then the *same* algorithm yields their greatest common divisor. This observation appears to be the contrapositive of the first, but Euclid distinguishes it as Proposition VII.2 of the *Elements.*

Conversational implicature may be seen in Apollonius's treating of the circle as different from an ellipse.

6. Lines

In the old understanding, a **line** need not be straight. A line may have endpoints, or it may be, for example, the circumference of a *circle.* Indeed, according to the definition in Euclid's *Elements,*

> A **circle** (*κύκλος*) is a plane figure contained by one line (*γραμμή*) such that all the straight lines falling upon it from one point among those lying within the figure are equal to one another.

A *straight* line (*εὐθεῖα γραμμή*) does have endpoints; but the straight line may be *produced* (extended) beyond these endpoints, as far as desired.

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Mathematics Dept, Middle East Technical University, Ankara 06531, Turkey
URL: `www.metu.edu.tr/~dpierce/`
E-mail address: `dpierce@metu.edu.tr`