# Differential equations notes

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I originally prepared these notes while teaching Math 253, Introduction to Ordinary Differential Equations, in the fall of 2003. Now I am updating them while teaching Math 219, Introduction to Differential Equations.

The main reference is [1, 7th edition, 2001], though sometimes I consult books like [2] or [3], used at METU years ago.

## References

[1] William E. Boyce and Richard C. DiPrima. *Elementary differential equations and boundary value problems*. John Wiley & Sons Inc., New York, 1965.

[2] Lyman M. Kells. *Elementary Differential Equations*. McGraw-Hill Book Company, New York, fifth edition, 1960.

[3] Shepley L. Ross. *Differential equations*. Blaisdell Publishing Company Ginn and Co. New York-Toronto-London, 1964.

[4] Michael Spivak. *Calculus*. Publish or Perish, Berkeley, California, 2nd edition, 1980.

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# o Introduction

Recall the fundamental theorems of calculus:

(1) If $f$ is continuous, then $\dfrac{\mathrm{d}}{\mathrm{d}\,x}\displaystyle\int_a^x f(t)\,\mathrm{d}\,t = f(x)$.

(2) If $f = g'$, then $\int_a^b f(x)\,\mathrm{d}\,x = g(b) - g(a)$.

So anybody who has studied calculus has solved differential equations:

- To find an indefinite integral $\int f(t)\,\mathrm{d}\,t$ is to solve the differential equation

$$y' = f(x). \tag{0.1}$$

  For example, since

$$\int 4t \ln t\,\mathrm{d}\,t = 2t^2 \ln t - \int 2t\,\mathrm{d}\,t = 2t^2 \ln t - t^2 + C, \tag{0.2}$$

  the solutions of the differential equation

$$y' = 4x \ln x \tag{0.3}$$

  are the functions $2x^2 \ln x - x^2 + C$.

- The value of $\ln x$ is defined to be $\int_1^x \mathrm{d}\,t/t$. Thus ln is the solution of the differential equation

$$y' = \frac{1}{x} \tag{0.4}$$

  meeting the *initial condition* $y(1) = 0$.

- The inverse of ln is defined to be exp, and we must have

$$\exp' x = \frac{1}{\ln' \exp x} = \exp x. \tag{0.5}$$

  Thus exp is the solution of the differential equation

$$y' = y \tag{0.6}$$

  meeting the initial condition $y(0) = 1$.

Note that equation (0.4) expresses the derivative of a function of $x$ as another function of $x$; so the equation can be solved by integrating. But in equation (0.6), the derivative of a function $y$ is expressed as a function *of that function* $y$; the solution of such equations will require new techniques, which are what this course is all about.

## A note on notation

Particular functions can be described in several ways. For example, the squaring function is:

- the function $f$ such that $f(x) = x^2$ for all [real] numbers $x$;

- the function $x \mapsto x^2$ (or $t \mapsto t^2$); or
- the function $x^2$ (or $t^2$).

(See [4, ch. 3, p. 45].) The last notation is ambiguous. The expressions $x \mapsto x^2$ and $t \mapsto t^2$ represent the *same* function; but we cannot indicate this by writing

$$x^2 = t^2. \tag{0.7}$$

Also, what does an expression like

$$t + t^2 x \tag{0.8}$$

mean? If it indicates a function, then this function could be

$$t \mapsto t + t^2 x, \quad \text{or} \quad x \mapsto t + t^2 x, \quad \text{or possibly} \quad (t, x) \mapsto t + t^2 x. \tag{0.9}$$

The definite integral

$$\int_a^b f(t)\,\mathrm{d}\,t \tag{0.10}$$

can also be written

$$\int_a^b f. \tag{0.11}$$

Of course the latter notation can't be used if we only know $f$ from a formula: the integral from $a$ to $b$ of the function $x \mapsto x \ln x$ is

$$\int_a^b x \ln x\,\mathrm{d}\,x; \tag{0.12}$$

this is a number (depending on $a$ and $b$), and it is the *same* number as

$$\int_a^b t \ln t\,\mathrm{d}\,t. \tag{0.13}$$

The so-called indefinite integral $\int f(t)\,\mathrm{d}\,t$ is usually understood as an instruction or command, namely:

(1) find a function $g$ such that $g' = f$;

(2) write down $g(t) + C$, where $C$ is understood as an arbitrary constant.

(See [4, ch. 18, p. 337].) In this case, we would also write

$$\int f(s)\,\mathrm{d}\,s = g(s) + C, \tag{0.14}$$

which means that $\int f(t)\,\mathrm{d}\,t$ and $\int f(s)\,\mathrm{d}\,s$ indicate *different things*.

However, this convention is violated at [1, § 2.1, p. 33, equations (21–22)]. There, by integrating

$$\frac{\mathrm{d}}{\mathrm{d}\,t}(\mathrm{e}^{at} y) = \mathrm{e}^{at} g(t), \tag{0.15}$$

the writers conclude

$$\mathrm{e}^{at} y = \int \mathrm{e}^{as} g(s)\,\mathrm{d}\,s + c; \tag{0.16}$$

they feel the need to distinguish the 'integration variable,' now $s$, from the 'independent variable,' $t$. Yet if the integration can be performed, then, according to the usual convention, an expression $h(s)$ will be obtained. Then one just has to remember to replace $s$ with $t$ here.

To avoid ambiguity, it might be best to let expressions such as $\int f(t)\,\mathrm{d}\,t$ and $\int f(s)\,\mathrm{d}\,s$ and $\int f$ indicate the *same* thing, namely an arbitrary function $g$ such that $g' = f$. In other words, these expressions should simply denote a **primitive of** $f$. Suppose we want to indicate a *value* of $g$, such as $g(x)$. Keeping in mind the fundamental theorem of calculus, we can write, for example,

$$\int^x f. \tag{0.17}$$

Then the conclusion of (0.2) would be written

$$\int^x 4t \ln t\,\mathrm{d}\,t = 2x^2 \ln x - x^2 + C. \tag{0.18}$$

If we are trying to be precise in this way, then it seems that we should write (0.1) in the form

$$y' = f. \tag{0.19}$$

But this form does not accommodate the example (0.3); this could be written more precisely as

$$\frac{\mathrm{d}\,y}{\mathrm{d}\,x} = 4x \ln x. \tag{0.20}$$

One can formally 'solve' this as follows:

$$\frac{\mathrm{d}\,y}{\mathrm{d}\,x} = 4x \ln x \iff \mathrm{d}\,y = 4x \ln x\,\mathrm{d}\,x \tag{0.21}$$

$$\iff \int \mathrm{d}\,y = \int 4x \ln x\,\mathrm{d}\,x \tag{0.22}$$

$$\iff y = 2x^2 \ln x - x^2 + C. \tag{0.23}$$

The *chain rule* is that if $g$ is differentiable at $a$, and $f$ is differentiable at $g(a)$, then $f \circ g$ is differentiable at $a$, and

$$(f \circ g)'(a) = f'(g(a)) \cdot g'(a). \tag{0.24}$$

Hence the equation

$$(f \circ g)' = (f' \circ g) \cdot g' \tag{0.25}$$

holds on the common domain of the two members. If one thinks of $f \circ g$ as a function $y$, and of $g$ as a function $x$, then one may write

$$\frac{\mathrm{d}\,y}{\mathrm{d}\,t} = \frac{\mathrm{d}\,y}{\mathrm{d}\,x}\frac{\mathrm{d}\,x}{\mathrm{d}\,t}. \tag{0.26}$$

Then $\mathrm{d}\,y/\,\mathrm{d}\,x$ is not the derivative $y'$ of $y$, but is the quotient of $y'$ by $x'$.

# 1 Examples from natural science

## 1.1 From physics

In physics, Newton's second law, $F = ma$, is a differential equation since $a = \mathrm{d}\,v/\,\mathrm{d}\,t$, that is, $a = \dot{v}$ (the time-derivative of $v$).

On a body falling in air, there are two forces:

- $mg$, due to gravity;
- $-\gamma v$, due to friction. (Here $\gamma$ is the 'drag-coefficient'.)

Then $ma = F$ can be written as $m\dot{v} = mg - \gamma v$, that is,

$$\frac{\mathrm{d}\,v}{\mathrm{d}\,t} = g - \frac{\gamma}{m}v. \tag{1.1}$$

## 1.2 From ecology

A similar equation arises from consideration of a mouse population of size $p$ attacked by owls. Changes in $p$ are due to:

- the birth-rate $r$, in units $\dfrac{\text{baby mice}}{(\text{mice})(\text{month})}$;
- the death-rate $k$, in units $\dfrac{\text{mice}}{\text{month}}$ (killed by owls).

(Note how 'rate' is used in two different senses.) Hence we have the differential equation

$$\frac{\mathrm{d}\,p}{\mathrm{d}\,t} = rp - k. \tag{1.2}$$

# 2 Direction-fields

The equations (1.1) and (1.2) have the form

$$\frac{\mathrm{d}\,y}{\mathrm{d}\,t} = ay - b, \tag{2.1}$$

where $y$ is a function of $t$, and $a$ and $b$ are constant. In turn, the equation (2.1) (along with (0.4) and (0.6)) is a special case of an equation

$$\frac{\mathrm{d}\,y}{\mathrm{d}\,t} = f(t, y), \tag{2.2}$$

where $f$ is a function of two variables. A solution to such an equation can be thought of as a curve in the Cartesian plane whose slope at each point $(t, y)$ has the value $f(t, y)$.

Hence an idea of the solution of (2.2) can be got from a **direction-field**. Formally, the direction-field associated with (2.2) is just the function $f$ itself. One

pictures the direction-field as the plane with a little line of slope $f(t, y)$ at each point $(t, y)$. See figures 1.1.2 and 1.1.4 in the book.

From (1.1) (and the direction-field) one may see that, as $t \to \infty$, so $v$ approaches a 'terminal velocity', $mg/\gamma$.

From (1.2), one may see that, as $t \to \infty$, so $p$ either drops to 0 or blows up (increases without bound).

# 3 First-order ordinary differential equations

## 3.1 Introduction

Equation (2.1) is an **ordinary** differential equation because $y$ is a function of only one variable, so only ordinary derivatives (and not partial derivatives) appear. Equation (2.1) is also:

- **first-order**, because only the first derivative of $y$ appears;
- **linear**, because $y$ and its derivative(s) are not raised to powers or multiplied by each other;
- with **constant coefficients**, because $a$ and $b$ are constant (that is, the derivatives $\mathrm{d}\,a/\,\mathrm{d}\,t$ and $\mathrm{d}\,b/\,\mathrm{d}\,t$ are 0).

To solve (2.1), assuming $a \neq 0$, first note that *one* solution is constant: $y = b/a$. To find all other solutions, we may proceed thus:

$$\frac{\mathrm{d}\,y}{\mathrm{d}\,t} = ay - b \iff \frac{\mathrm{d}\,y}{\mathrm{d}\,t} = a\left(y - \frac{b}{a}\right) \tag{3.1}$$

$$\iff \frac{\mathrm{d}\,y/\,\mathrm{d}\,t}{y - b/a} = a \tag{3.2}$$

$$\iff \frac{\mathrm{d}}{\mathrm{d}\,t} \ln\left|y - \frac{b}{a}\right| = a \tag{3.3}$$

$$\iff \ln\left|y - \frac{b}{a}\right| = at + C \tag{3.4}$$

$$\iff \left|y - \frac{b}{a}\right| = e^{at}e^{C} \tag{3.5}$$

$$\iff y = \frac{b}{a} + ce^{at}. \tag{3.6}$$

Note that $c = \pm e^{C}$; but what does this constant mean? Suppose $y(0)$ is $y_0$—this is an **initial condition**. Then $y_0 = b/a + c$, so $c = y_0 - b/a$. Hence the solution of (2.1) is

$$y = \frac{b}{a} + \left(y_0 - \frac{b}{a}\right)e^{at}. \tag{3.7}$$

For another method of solution, see Exercise 1.2.4.

## 3.2 Integrating factors

For a *third* method of solving (2.1), rewrite the equation as

$$\frac{\mathrm{d}\,y}{\mathrm{d}\,t} - ay = -b. \tag{3.8}$$

Now we use a trick: Multiply each side by a function $\mu(t)$ (called an **integrating factor**):

$$\frac{\mathrm{d}\,y}{\mathrm{d}\,t} \cdot \mu(t) - ay \cdot \mu(t) = -b \cdot \mu(t). \tag{3.9}$$

Try to understand the left side as a derivative. We know

$$\frac{\mathrm{d}}{\mathrm{d}\,t}(y \cdot \mu(t)) = \frac{\mathrm{d}\,y}{\mathrm{d}\,t} \cdot \mu(t) + y \cdot \mu'(t). \tag{3.10}$$

So let us choose $\mu$ so that

$$-a \cdot \mu(t) = \mu'(t). \tag{3.11}$$

This is a new differential equation. We just need one solution, say $\mu(t) = e^{-at}$. Substituting into (3.9) and integrating, we have

$$\frac{\mathrm{d}\,y}{\mathrm{d}\,t} = ay - b \iff \frac{\mathrm{d}\,y}{\mathrm{d}\,t} \cdot e^{-at} - ay \cdot e^{-at} = -be^{-at} \tag{3.12}$$

$$\iff \frac{\mathrm{d}}{\mathrm{d}\,t}(y \cdot e^{-at}) = -be^{-at} \tag{3.13}$$

$$\iff y \cdot e^{-at} = \frac{b}{a}e^{-at} + c \tag{3.14}$$

$$\iff y = \frac{b}{a} + ce^{at}, \tag{3.15}$$

in agreement with (3.6).

Now: in arriving at (3.13), we had no need to assume that $b$ was constant. Replacing $-b$ with $g(t)$, we have

$$\frac{\mathrm{d}\,y}{\mathrm{d}\,t} - ay = g(t) \iff \frac{\mathrm{d}}{\mathrm{d}\,t}(y \cdot e^{-at}) = g(t)e^{-at} \tag{3.16}$$

$$\iff y \cdot e^{-at} = \int^t g(s)e^{-as}\,\mathrm{d}\,s \tag{3.17}$$

$$\iff y = e^{at}\int^t g(s)e^{-as}\,\mathrm{d}\,s. \tag{3.18}$$

(For the notation, see (0.17).) If $g$ is a polynomial, then we can find $\int g(t)e^{-at}\,\mathrm{d}\,t$ through integration by parts.

**3.1 Example.** Say $g(t) = -(mt + k)$. We have

$$\int (mt+k)e^{-at}\,\mathrm{d}\,t = -\frac{mt+k}{a}e^{-at} + \frac{m}{a}\int e^{-at}\,\mathrm{d}\,t \tag{3.19}$$

$$= -\left(\frac{mt+k}{a} + \frac{m}{a^2}\right)e^{-at} + C. \tag{3.20}$$

Therefore

$$\frac{\mathrm{d}\,y}{\mathrm{d}\,t} - ay = -(mt+k) \iff y = \frac{mt+k}{a} + \frac{m}{a^2} + Ce^{at}. \tag{3.21}$$

Note here that the behavior of $y$ for large $t$ depends on the sign of $a$:

- if $a \leqslant 0$, then $y \approx (mt + k)/a + m/a^2$ as $t \to \infty$;
- if $a > 0$ and $C \neq 0$, then $y$ diverges from this value.

We can try using integrating factors to solve an arbitrary first-order linear equation, namely

$$\frac{\mathrm{d}\, y}{\mathrm{d}\, t} + y \cdot p(t) = g(t). \tag{3.22}$$

Our integrating factor $\mu(t)$ should satisfy

$$p(t)\mu(t) = \mu'(t), \tag{3.23}$$

$$\frac{\mu'(t)}{\mu(t)} = p(t), \tag{3.24}$$

$$\frac{\mathrm{d}}{\mathrm{d}\, t} \ln \mu(t) = p(t). \tag{3.25}$$

So we may let

$$\ln \mu(t) = \int^t p, \tag{3.26}$$

$$\mu(t) = \exp \int^t p. \tag{3.27}$$

For this $\mu$, we have

$$\frac{\mathrm{d}\, y}{\mathrm{d}\, t} + y \cdot p(t) = g(t) \iff \frac{\mathrm{d}}{\mathrm{d}\, t}(y \cdot \mu(t)) = g(t) \cdot \mu(t) \tag{3.28}$$

$$\iff y \cdot \mu = \int g \cdot \mu. \tag{3.29}$$

How useful this is depends on whether we can perform the integrations:

**3.2 Example.** Suppose we want to solve

$$ty' + ay = bt^2, \tag{3.30}$$

that is,

$$y' + y \cdot \frac{a}{t} = bt. \tag{3.31}$$

To find $\mu$ as above, we note

$$\int^t \frac{a}{s}\, \mathrm{d}\, s = a \ln t + C, \tag{3.32}$$

so we may let

$$\mu(t) = \exp(a \ln t) = t^a. \tag{3.33}$$

Then we have

$$y' + y \cdot \frac{a}{t} = bt \iff \frac{\mathrm{d}}{\mathrm{d}\, t}(y \cdot t^a) = bt^{a+1} \tag{3.34}$$

$$\iff y \cdot t^a = \frac{b}{a+2} t^{a+2} + C \tag{3.35}$$

$$\iff y = \frac{b}{a+2} t^2 + Ct^{-a}. \tag{3.36}$$

We must assume here $a \neq -2$.

## 3.3 Separable equations

We can use the technique of integrating factors to attack an equation

$$\frac{\mathrm{d}\,y}{\mathrm{d}\,x} = f(x,y), \tag{3.37}$$

provided that $f(x,y)$ is linear in $y$; in this case we solve as we did (3.22). Suppose now that in $f(x,y)$, we can *separate* the two variables in the sense of putting $x$ in the numerator and $y$ in the denominator, so that (3.37) becomes

$$\frac{\mathrm{d}\,y}{\mathrm{d}\,x} = \frac{g(x)}{h(y)}. \tag{3.38}$$

The notation suggests the following solution:

$$\frac{\mathrm{d}\,y}{\mathrm{d}\,x} = \frac{g(x)}{h(y)} \iff h(y)\,\mathrm{d}\,y = g(x)\,\mathrm{d}\,x \tag{3.39}$$

$$\iff \int h(y)\,\mathrm{d}\,y = \int g(x)\,\mathrm{d}\,x. \tag{3.40}$$

If in fact $h = H'$ and $g = G'$, then (3.40) is equivalent to

$$H(y) = G(x) + c. \tag{3.41}$$

This equation describes a family of curves, one for each value of $c$. Each curve determines $y$ *implicitly* as one or more functions of $x$. Perhaps we want the curve to contain the point $(x_0, y_0)$; then we must have

$$H(y_0) = G(x_0) + c, \tag{3.42}$$

which lets us rewrite (3.41) as

$$H(y) - H(y_0) = G(x) - G(x_0), \tag{3.43}$$

which in turn means that (3.40) can be written

$$\int_{y_0}^{y} h(t)\,\mathrm{d}\,t = \int_{x_0}^{x} g(t)\,\mathrm{d}\,t. \tag{3.44}$$

**3.3 Example.** To solve

$$\frac{\mathrm{d}\,y}{\mathrm{d}\,x} = \frac{1-x^2}{y^2} \tag{3.45}$$

so that $y(0) = 3$, we can rewrite the requirements as

$$\int_3^y t^2\,\mathrm{d}\,t = \int_0^x (1-t^2)\,\mathrm{d}\,t, \tag{3.46}$$

$$\frac{y^3}{3} - 9 = x - \frac{x^3}{3}, \tag{3.47}$$

$$y^3 = 27 + 3x - x^3. \tag{3.48}$$

Since every real number has a unique real cube-root, (3.48) determines $y$ as a unique function of $x$. One can get a sense for the shape of the function by looking at the direction-field...

The technique can be justified more formally as follows. Rewrite (3.38) as

$$y' = \frac{g}{h(y)}, \tag{3.49}$$

where $y$, $g$ and $h$ are understood to be functions. Equivalently,

$$h(y)y' = g, \tag{3.50}$$

$$\int_{x_0}^{x} h(y)y' = \int_{x_0}^{x} g. \tag{3.51}$$

If $y(x_0) = y_0$, then by so-called $u$-substitution (here $u = y(t)$) we can write (3.51) as

$$\int_{y_0}^{y(x)} h = \int_{x_0}^{x} g, \tag{3.52}$$

which is (3.44).

Note that linear differential equations with no constant term are separable:
**3.4 Example.** To solve the equation

$$y' = y + y\sin x \tag{3.53}$$

so that $y(0) = 1$, we may note

$$y' = y + y\sin x \iff y' = y(1+\sin x) = \frac{1+\sin x}{1/y} \tag{3.54}$$

$$\iff \int_1^y \frac{\mathrm{d}\,t}{t} = \int_0^x (1+\sin t)\,\mathrm{d}\,t \tag{3.55}$$

$$\iff \ln y = (t - \cos t)|_0^x = x - \cos x + 1 \tag{3.56}$$

$$\iff y = \exp(x - \cos x + 1). \tag{3.57}$$

A separable equation with initial condition may not have a unique solution:
**3.5 Example.** To solve

$$\frac{\mathrm{d}\,y}{\mathrm{d}\,x} = -\frac{x}{y} \quad \text{and} \quad y(1) = 0, \tag{3.58}$$

we note that the solutions are just those $y$ satisfying:

$$\int_0^y t\,\mathrm{d}\,t = -\int_1^x t\,\mathrm{d}\,t, \tag{3.59}$$

$$\left.\frac{t^2}{2}\right|_0^y = -\left.\frac{t^2}{2}\right|_1^t, \tag{3.60}$$

$$y^2 = -x^2 + 1, \tag{3.61}$$

$$y = \pm\sqrt{x^2+1}. \tag{3.62}$$

We now have three methods of solution for some equations:
**3.6 Example.** We can solve

$$y' = f(t)\cdot(y+a), \tag{3.63}$$

where $a$ is constant, in any of the following ways:

(1) Rewrite as follows:

$$\frac{y'}{y+a} = f(t), \tag{3.64}$$

$$\frac{\mathrm{d}}{\mathrm{d}\,t} \ln|y+a| = f(t), \tag{3.65}$$

$$\ln|y+a| = \int f + C, \tag{3.66}$$

$$|y+a| = \exp C \exp \int f, \tag{3.67}$$

$$y+a = c \exp \int f, \tag{3.68}$$

$$y = c \exp \int f - a. \tag{3.69}$$

In particular,

$$y_0 = c \exp \int_{t_0}^{t_0} f - a = c - a, \tag{3.70}$$

so the solution is

$$y = (a+y_0) \exp \int_{t_0}^{t} f - a. \tag{3.71}$$

(2) The equation is separable:

$$\frac{\mathrm{d}\,y}{\mathrm{d}\,t} = \frac{f(t)}{(y+a)^{-1}}, \tag{3.72}$$

$$\int_{y_0}^{y} \frac{\mathrm{d}\,x}{x+a} = \int_{t_0}^{t} f(x)\,\mathrm{d}\,x, \tag{3.73}$$

$$\ln \pm(y+a) - \ln \pm(y_0+a) = \int_{t_0}^{t} f, \tag{3.74}$$

which is equivalent to (3.71).

(3) We can use integrating factors; but watch out: We want to solve

$$\frac{\mathrm{d}\,y}{\mathrm{d}\,t} \cdot \mu(t) - y \cdot f(t) \cdot \mu(t) = a \cdot f(t) \cdot \mu(t), \tag{3.75}$$

where the left side is

$$\frac{\mathrm{d}}{\mathrm{d}\,t}(y \cdot \mu(t)), \tag{3.76}$$

so we want

$$\mu'(t) = -f(t) \cdot \mu(t), \tag{3.77}$$

$$\frac{\mu'(t)}{\mu(t)} = -f(t), \tag{3.78}$$

$$\frac{\mathrm{d}}{\mathrm{d}\,t} \ln|\mu(t)| = -f(t), \tag{3.79}$$

$$\ln|\mu(t)| = -\int^{t} f. \tag{3.80}$$

Any $\mu$ satisfying this will work, so let

$$\mu(t) = \exp \int^t f. \tag{3.81}$$

Then we have

$$\frac{\mathrm{d}}{\mathrm{d}\,t}(y \cdot \mu(t)) = a \cdot f(t) \cdot \mu(t) = -a \cdot \mu'(t), \tag{3.82}$$

$$\int_{t_0}^t \frac{\mathrm{d}}{\mathrm{d}\,u}(y \cdot \mu(u))\,\mathrm{d}\,u = \int_{t_0}^t \mu'(u)\,\mathrm{d}\,u. \tag{3.83}$$

If we assume $\mu(t_0) = 0$, then we get

$$y \cdot \mu(t) = -a \cdot \mu(t), \tag{3.84}$$

$$y = -a, \tag{3.85}$$

which is only a special case of the earlier solutions. But because of its definition here, $\mu(t)$ can never be 0. We may specify

$$\mu(t) = \exp \int_{t_0}^t f, \tag{3.86}$$

so $\mu(t_0) = 1$. Then (3.84) gives us

$$t \cdot \mu(t) - y_0 = -a(\mu(t) - 1), \tag{3.87}$$

$$y = \frac{a + y_0}{\mu(t)} - a = (a + y_0) \exp \int_{t_0}^t f - a, \tag{3.88}$$

as before.

**3.7 Example.** We find an expression for the velocity $v$ of a projectile fired vertically from an airless planet. We may let $x$ be the distance of the projectile from the center of the planet. From Newton we have

$$\frac{\mathrm{d}\,v}{\mathrm{d}\,t} = -\frac{k}{x^2} \tag{3.89}$$

for some positive constant $k$ that is proportional to the planet's mass.

Either the projectile eventually returns to the planet, or not.

During certain time-intervals, '$v$ is a function of $x$,' that is, $v = f \circ x$ for some function $f$; so by the Chain Rule (see p. 3),

$$\frac{\mathrm{d}\,v}{\mathrm{d}\,t} = \frac{\mathrm{d}\,v}{\mathrm{d}\,x}\frac{\mathrm{d}\,x}{\mathrm{d}\,t} = v \cdot \frac{\mathrm{d}\,v}{\mathrm{d}\,x}. \tag{3.90}$$

Thus we get a separable equation:

$$v \cdot \frac{\mathrm{d}\,v}{\mathrm{d}\,x} = -\frac{k}{x^2}, \tag{3.91}$$

$$\int_{v_0}^v u\,\mathrm{d}\,u = -k \int_{x_0}^x \frac{\mathrm{d}\,u}{u^2}, \tag{3.92}$$

$$v^2 - v_0^2 = 2k\left(\frac{1}{x} - \frac{1}{x_0}\right), \tag{3.93}$$

$$v^2 = \frac{2k}{x} + v_0^2 - \frac{2k}{x_0}. \tag{3.94}$$

The limiting case is $v_0^2 = 2k/x_0$; the projectile returns if and only if

$$v_0 \leqslant \sqrt{\frac{2k}{x_0}}. \tag{3.95}$$

**3.8 Example.** [I didn't do this in class.] Say a container (a bucket, a pond) of volume $V$ is fed at a constant rate $a$ (say in liters/minute) with a solution of some chemical compound with a variable concentration of $f$ (in grams/liter, say). The same container is drained at the same rate. Assume that at any moment the concentration of compound in the container is uniform. Let the total mass of the compound in the container be $Q$. Then we get

$$Q' = af(t) - a\frac{Q}{V}, \tag{3.96}$$

a linear equation soluble by means of the integrating factor $\exp(at/V)$:

$$a \cdot f(t) \cdot \exp\frac{at}{V} = Q'(t) \cdot \exp\frac{at}{V} + \frac{a}{V}Q(t)\exp\frac{at}{V} = \frac{\mathrm{d}}{\mathrm{d}\,t}\left(Q'(t) \cdot \exp\frac{at}{V}\right), \tag{3.97}$$

$$a\int f(t)\exp\frac{at}{V}\,\mathrm{d}\,t + C. \tag{3.98}$$

Say $t_0 = 0$ and $Q(t_0) = Q_0$. Then

$$Q(t) = \frac{\displaystyle\int_0^t f(u)\exp\frac{au}{V}\,\mathrm{d}\,u}{\exp\dfrac{at}{V}} + Q_0. \tag{3.99}$$

## 3.4 Linear *versus* non-linear equations

Here is precise formulation of the method of integrating factors as applied to linear differential equations:

**3.9 Theorem.** *Suppose $p$ and $g$ are continuous functions on the open interval $(\alpha, \beta)$, and $\alpha < t_0 < \beta$. Then for any real number $y_0$ there is a unique function $y$ that is differentiable on $(\alpha, \beta)$ and solves the initial-value problem*

$$y' + y \cdot p = g, \tag{3.100}$$

$$y(t_0) = y_0. \tag{3.101}$$

*Proof.* Let $\mu$ be the function on $(\alpha, \beta)$ given by

$$\mu(t) = \exp\int_{t_0}^t p. \tag{3.102}$$

Note that $\mu$ is well-defined on $(\alpha, \beta)$ since $p$ is continuous. Also,

$$\mu' = p \cdot \mu, \tag{3.103}$$

$\mu(t_0) = 1$ and $\mu(t)$ is never 0. Suppose (3.100) and (3.101) hold. Then we have

$$y' \cdot \mu + y \cdot p \cdot \mu = g \cdot \mu, \tag{3.104}$$

$$(y \cdot \mu)' = g \cdot \mu, \tag{3.105}$$

$$\int_{t_0}^{t} (y \cdot \mu)' = \int_{t_0}^{t} g \cdot \mu, \tag{3.106}$$

$$y(t)\mu(t) - y_0 = \int_{t_0}^{t} g \cdot \mu, \tag{3.107}$$

$$y(t) = \frac{\int_{t_0}^{t} g \cdot \mu}{\mu(t)} + \frac{y_0}{\mu(t)} \tag{3.108}$$

for all $t$ in $(\alpha, \beta)$. Thus $y$ is unique. Multiplying by $\mu(t)$ and differentiating, one can check that the value of $y$ given by (3.108) does satisfy (3.100) and (3.101). □

**3.10 Example.** In Example 3.2, we consider the equation

$$ty' + ay = bt^2. \tag{3.109}$$

The corresponding linear equation is

$$y' + y \cdot \frac{a}{t} = bt. \tag{3.110}$$

Then Theorem 3.9 tells that, if $(t_0, y_0)$ is a point in the plane such that $t_0 =\neq 0$, then (3.109) has a unique solution such that $y(t_0) = y_0$, and $y$ is defined on: $(0, \infty)$, if $t_0 > 0$, and on $(-\infty, 0)$, if $t_0 < 0$. If $a \neq -2$, we found

$$y = \frac{b}{a+2} t^2 + Ct^{-a}, \tag{3.111}$$

from which we get

$$y_0 = \frac{b}{a+2} t_0^2 + Ct_0^{-a}, \tag{3.112}$$

$$C = y_0 t_0^a - \frac{b}{a+2} t_0^{a+2}, \tag{3.113}$$

$$y = \frac{b}{a+2} t^2 + y_0 \left(\frac{t_0}{t}\right)^a - \frac{b}{a+2} \left(\frac{t_0}{t}\right)^a t_0^2; \tag{3.114}$$

this is the unique solution. But say $a = -1$; then

$$y = bt^2 + \frac{y_0}{t_0} t - bt_0 t. \tag{3.115}$$

Thus $y$ is well-defined on $\mathbb{R}$ (and is a solution of (3.109)).

Does a theorem like Theorem 3.9 hold for an arbitrary first-order equation,

$$y'(t) = f(t, y(t))? \tag{3.116}$$

In time we shall give some justification for the following assertion:

*Suppose a rectangle $(\alpha, \beta) \times (\gamma, \delta)$ contains a point $(t_0, y_0)$, and on this rectangle, a function $f$ is continuous and has a continuous partial derivative in the second coordinate. Then the initial-value problem*

$$y'(t) = f(t, y(t)), \tag{3.117}$$
$$y(t_0) = y_0 \tag{3.118}$$

*has a unique solution on* some *interval containing $t_0$.*

To see why this might be true, consider the direction-field for the equation. Note how continuity matters: For example, in Example 3.5, we have an initial-value problem $\mathrm{d}\,y/\,\mathrm{d}\,x = -x/y$ and $y(1) = 0$ with *two* solutions; but the function

$$(x, y) \mapsto -\frac{x}{y} \tag{3.119}$$

is not defined (much less continuous) at $(0, 1)$.

We also have the following:

**3.11 Example.** Is there an initial-value problem whose solutions are the curves $y = at^2$? These pass through $(0, 0)$, and on such curves we have

$$y' = 2at, \tag{3.120}$$
$$a = \frac{y}{t^2}, \tag{3.121}$$
$$y' = \frac{2y}{t}, \tag{3.122}$$
$$y't = 2y. \tag{3.123}$$

The direction-field is undefined at $(0, 0)$, so the proposed theorem does not apply. We can still solve the equation:

$$\frac{\mathrm{d}\,y}{2y} = \frac{\mathrm{d}\,t}{t}, \tag{3.124}$$
$$\frac{1}{2}(\ln y - \ln y_0 = \ln t - \ln t_0, \tag{3.125}$$
$$\ln y + 2\ln t_0 = 2\ln t + \ln y_0, \tag{3.126}$$
$$yt_0^2 = t^2 y_0. \tag{3.127}$$

**3.12 Example.** Assume $y_0 \neq 0$. If we solve

$$y' = y^2 \tag{3.128}$$

as a separable equation, we get

$$t - t_0 = \int_{y_0}^{y} u^{-2}\,\mathrm{d}\,u = \frac{1}{y_0} - \frac{1}{y}, \tag{3.129}$$
$$y = \frac{y_0}{1 - (t - t_0)y_0}. \tag{3.130}$$

Then $y$ is undefined when $t = t_0 + 1/y_0$, something not evident from the original equation (which meets the criteria of the proposed theorem on all rectangles). The solution should be understood as defined on: $(-\infty, t_0 + 1/y_0)$, if $y_0 > 0$; and on $(t_0 + 1/y_0, \infty)$, if $y_0 < 0$. If we require $y_0 = 0$, then, by inspection, the equation has the solution $y = 0$, defined on $(-\infty, \infty)$; by the proposed theorem, the solution is unique.

## 3.5 Exact equations

We ask which first-order equations $y' = f(x, y)$ have an implicit solution

$$\psi(x, y) = 0. \tag{3.131}$$

Differentiating the solution, we get

$$\psi_x(x, y) + \psi_y(x, y) \cdot y' = 0. \tag{3.132}$$

(Note that $\psi_x$ is just $\partial\psi/\partial x$, &c.) Such an equation is called **exact**, and its solutions have the form

$$\psi(x, y) = C. \tag{3.133}$$

Note that a multiple

$$\mu(x, y) \cdot \psi_x(x, y) + \mu(x, y) \cdot \psi_y(x, y) \cdot y' = 0 \tag{3.134}$$

of an exact equation will not generally be exact, although it is solved by all solutions of the exact equation.

**3.13 Example.** The separable equation

$$f(x) + g(y) \cdot y' = 0 \tag{3.135}$$

is exact, since it has the form (3.132) with

$$\psi(x, y) = \int f(x) \,\mathrm{d}\, x + \int g(y) \,\mathrm{d}\, y. \tag{3.136}$$

**3.14 Example.** The linear equation

$$y' + y \cdot g(x) = p(x) \tag{3.137}$$

is not generally exact, but if $\mu'(x) = g(x) \cdot \mu(x)$, then we can rewrite the equation as

$$y \cdot \mu'(x) - p(x) \cdot \mu(x) + \mu(x) \cdot y' = 0, \tag{3.138}$$

which has the form (3.132) with

$$\psi(x, y) = y \cdot \mu(x) - \int p(x) \cdot \mu(x) \,\mathrm{d}\, x. \tag{3.139}$$

**3.15 Example.** The equation

$$2x + y^2 + 2xy \cdot y' = 0 \tag{3.140}$$

has the form of (3.132), where

$$\psi(x, y) = x^2 + xy^2. \tag{3.141}$$

How can you tell whether an equation of the form

$$M(x, y) + N(x, y) \cdot y' = 0 \tag{3.142}$$

is exact?

**3.16 Lemma.** *Let $M$ and $N$ be continuous functions of $(x, y)$ on an open rectangle $R$, and suppose $M_y$ and $N_x$ exist and are continuous on that rectangle. The following are equivalent:*

*(1) There is a function $\psi$ such that $\psi_x = M$ and $\psi_y = N$ on $R$.*

*(2) $M_y = N_x$ on $R$.*

*Proof.* If $\psi$ exists, then $\psi_{xy} = M_y$ and $\psi_{yx} = N_x$; since these derivatives are continuous, they are equal.

If $M_y = N_x$, we recover $\psi$ by integrating: Say

$$\psi(x, y) = \int M(x, y)\,\mathrm{d}\,x + h(y) \tag{3.143}$$

for some function $h$. Then

$$\psi_x = M. \tag{3.144}$$

We want to choose $h$ so that $\psi_y = N$. We have

$$\psi_y = \int M_y(x, y)\,\mathrm{d}\,x + h'(y), \tag{3.145}$$

$$h'(y) = \psi_y - \int M_y(x, y)\,\mathrm{d}\,x. \tag{3.146}$$

We want the right member of this equation to be $N - \int M_y(x, y)\,\mathrm{d}\,x$. It is enough if this function is constant in $x$, as $h'(y)$ should be: that is, it is enough if

$$0 = \frac{\partial}{\partial x}\left(N - \int M_y(x, y)\,\mathrm{d}\,x\right) = N_x - M_y, \tag{3.147}$$

which is true by assumption. Hence we may choose

$$h = \int \left(N - \int M_y(x, y)\,\mathrm{d}\,x\right)\mathrm{d}\,y. \tag{3.148}$$

□

**3.17 Example.** To solve

$$2xe^y - y\sin x + (x^2e^y + \cos x - 1)y' = 0, \tag{3.149}$$

we first check for exactness:

$$\frac{\partial}{\partial y}(2xe^y - y\sin x) = 2xe^y - \sin x, \tag{3.150}$$

$$\frac{\partial}{\partial x}(x^2e^y + \cos x - 1) = 2xe^y - \sin x, \tag{3.151}$$

equal, so the equation is exact. We now look for $\psi$ so that

$$\psi_x = 2xe^y - y\sin x, \tag{3.152}$$

$$\psi_y = x^2e^y + \cos x - 1. \tag{3.153}$$

One may be able to see that the solution is

$$x^2e^y + y\cos x - y = C. \tag{3.154}$$

Otherwise, find $\psi$ by integrating:

$$\psi = \int \psi_x \,\mathrm{d}\, x \tag{3.155}$$

$$= \int (2xe^y - y\sin x)\,\mathrm{d}\, x \tag{3.156}$$

$$= x^2e^y + y\cos x + h(y); \tag{3.157}$$

differentiating with respect to $y$ gives

$$x^2e^y + \cos x - 1 = \psi_y = x^2e^y + \cos x + h'(y), \tag{3.158}$$

$$h'(y) = -1, \tag{3.159}$$

$$h(y) = -y + C. \tag{3.160}$$

**3.18 Example.** The equation

$$y^2 + xy + (x^2 + 3xy)y' = 0 \tag{3.161}$$

is not exact. Perhaps we can find an *integrating factor*: some $\mu(x, y)$ such that the equation

$$\mu(x, y)\cdot(y^2 + xy) + \mu(x, y)\cdot(x^2 + 3xy)y' = 0 \tag{3.162}$$

is exact. Then we should have

$$\mu(x, y)\cdot(2y + x) + \mu_y(x, y)\cdot(y^2 + xy) = \mu(x, y)\cdot(2x + 3y) + \mu_x(x, y)\cdot(x^2 + 3xy), \tag{3.163}$$

a *partial differential equation*, which we are not prepared to solve in general. But perhaps there is a solution $\mu(x)$; that is, perhaps we can solve

$$\mu(x)\cdot(2y + x) = \mu(x)\cdot(2x + 3y) + \mu'(x)\cdot(x^2 + 3xy), \tag{3.164}$$

$$\mu'(x) = \mu(x)\frac{2y + x - (2x + 3y)}{x^2 + 3xy}, \tag{3.165}$$

$$\mu'(x) = -\mu(x)\frac{x + y}{x(x + 3y)}, \tag{3.166}$$

not soluble by current techniques. Try again: perhaps (3.163) has a solution $\mu(y)$, that is, we may be able to solve

$$\mu(y)\cdot(2y + x) + \mu'(y)\cdot(y^2 + xy) = \mu(y)\cdot(2x + 3y), \tag{3.167}$$

$$\mu'(y) = \mu(y)\frac{2x + 3y - (2y + x)}{y^2 + xy} = \mu(y)\frac{x + y}{y(x + y)} = \frac{\mu(y)}{y}, \tag{3.168}$$

so we may let $\mu(y) = y$. We now have to solve the exact equation

$$y^3 + xy^2 + (x^2y + 3xy^2)\cdot y' = 0. \tag{3.169}$$

The solution is $\psi(x,y) = C$, where

$$\psi = \int \psi_x(x,y)\,\mathrm{d}\,x = \int (y^3 + xy^2)\,\mathrm{d}\,x = xy^3 + \frac{1}{2}x^2y^2 + h(y), \tag{3.170}$$

$$x^2y + 3xy^2 = \psi_y = 3xy^2 + x^2y + h'(y), \tag{3.171}$$

$$h'(y) = 0, \tag{3.172}$$

so $h$ is constant and the solution is

$$xy^3 + \frac{1}{2}x^2y^2 = C. \tag{3.173}$$

# 4 Linear equations

An arbitrary linear ordinary differential equation can be written

$$y \cdot f_0(t) + y' \cdot f_1(t) + y'' \cdot f_2(t) + \cdots + y^{(n)} \cdot f_n(t) = g(t) \tag{4.1}$$

or, more neatly,

$$\sum_{i=0}^{n} y^{(i)} \cdot f_i(t) = g(t); \tag{4.2}$$

if $f_n \neq 0$, then this equation is of order $n$, and one can rewrite the equation as

$$\sum_{i=0}^{n-1} y^{(i)} \cdot \frac{f_i(t)}{f_n(t)} + y^{(n)} = \frac{g(t)}{f_n(t)}. \tag{4.3}$$

## 4.1 Homogeneous equations with constant coefficients, I

The equation (4.2) is **homogeneous** if $g = 0$. An arbitrary *second-order* linear homogeneous equation with constant coefficients is

$$y'' + ay' + b = 0. \tag{4.4}$$

**4.1 Example.** One may notice that the equation

$$y'' - y = 0 \tag{4.5}$$

has solutions $e^t$ and $e^{-t}$. Then, because of the sum- and constant-rules of calculus, every function

$$c_0e^t + c_1e^{-t} \tag{4.6}$$

is a solution (where the $c_i$ are constants). An initial condition such as

$$y(0) = 3 \tag{4.7}$$

gives us only

$$c_0 + c_1 = 3; \tag{4.8}$$

to have a unique solution, we can impose an initial condition on $y'$, such as

$$y'(0) = 2; \tag{4.9}$$

then we have also

$$c_0 - c_1 = 2. \tag{4.10}$$

The system of equations (4.8) and (4.10) yields

$$y = \frac{5}{2}e^t + \frac{1}{2}e^{-t} \tag{4.11}$$

as the unique solution of the form (4.6) to the initial-value problem.

We may hypothesize that (4.4) has a solution $e^{rt}$ for some $r$. Then we can check:

$$\frac{\mathrm{d}^2}{\mathrm{d}\,t^2}e^{rt} + a\frac{\mathrm{d}}{\mathrm{d}\,t}e^{rt} + be^{rt} = 0 \iff \tag{4.12}$$

$$(r^2 + ar + b)e^{rt} = 0 \iff \tag{4.13}$$

$$r^2 + ar + b = 0. \tag{4.14}$$

Equation (4.14) is called the **characteristic equation** of (4.4); it is a quadratic equation, with two or one real root, or else two or one complex, non-real root.

Suppose first that (4.14) has the two distinct real roots $r_0$ and $r_1$. Then, just as in Example 4.1, Equation (4.4) has the solutions

$$y = c_0e^{r_0t} + c_1e^{r_1t}. \tag{4.15}$$

This is called the **general solution**, since it will turn out that all solutions have this form. The initial conditions

$$y(t_0) = y_0 \quad \text{and} \quad y'(t_0) = y_0' \tag{4.16}$$

yield the linear system

$$\begin{cases} y_0 = c_0e^{r_0t_0} + c_1e^{r_1t_0}, \\ y_0' = r_0c_0e^{r_0t_0} + r_1c_1e^{r_1t_0}, \end{cases} \tag{4.17}$$

which has a unique solution $(c_0, c_1)$: We can check this just by solving the system, or by using theory:

If $a$, $b$, $c$ and $d$ are numbers, we write

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \tag{4.18}$$

for the number $ad - bc$; this is the **determinant** of the $2 \times 2$ **matrix**

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{4.19}$$

If this is not 0, then the linear system

$$\begin{cases} ax + by = e, \\ cx + dy = f \end{cases} \tag{4.20}$$

has the solution

$$x = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{ed - bf}{ad - bc}, \quad y = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{af - ec}{ad - bc}, \tag{4.21}$$

as one can check by computing. So (4.17) has the solution

$$c_0 = \frac{y_0 r_1 - y_0'}{r_1 - r_0} e^{-r_0 t_0}, \quad c_1 = \frac{y_0' - y_0 r_0}{r_1 - r_0} e^{-r_1 t_0}. \tag{4.22}$$

**4.2 Example.** For the general solution of

$$y'' + y' - 12y = 0, \tag{4.23}$$

form the characteristic equation,

$$r^2 + r - 12 = 0, \tag{4.24}$$

whose roots are $-4$ and 3; so the general solution is

$$y = c_0 e^{-4t} + c_1 e^{3t}. \tag{4.25}$$

Under the initial conditions

$$y(0) = 5 \quad \text{and} \quad y'(0) = -1, \tag{4.26}$$

we get the system

$$\begin{cases} 5 = \phantom{-}c_0 + \phantom{4}c_1, \\ -1 = -4c_0 + 3c_1, \end{cases} \tag{4.27}$$

with solution $(16/7, 19/7)$, whence

$$y = \frac{16}{7} e^{-4t} + \frac{19}{7} e^{3t}. \tag{4.28}$$

Note that the graph has a minimum point $(t_c, y_c)$, where

$$0 = y_c' = -\frac{64}{7} e^{-4t_c} + \frac{57}{7} e^{3t_c}, \tag{4.29}$$

$$e^{-7t_c} = \frac{57}{64}, \tag{4.30}$$

$$t_c = \frac{1}{7} \ln \frac{64}{57}. \tag{4.31}$$

Note how the long-term behavior of (4.15) depends on the signs of $r_0$ and $r_1$.

## 4.2 Homogeneous equations in general

An arbitrary second-order linear differential equation can be written

$$y'' + p(t)y' + q(t)y = g(t). \tag{4.32}$$

It can be proved that, *if the functions $p$, $q$ and $g$ are continuous on an open interval, then the equation has a unique solution on the whole interval satisfying initial conditions*

$$y_0 = y(t_0) \quad \text{and} \quad y_0' = y'(t_0). \tag{4.33}$$

We are interested now in the homogeneous case, where $g = 0$.

**4.3 Theorem** (principle of superposition)**.** *If $\phi_0$ and $\phi_1$ are solutions of*

$$y'' + p(t)y' + q(t)y = 0, \tag{4.34}$$

*and $c_0$ and $c_1$ are constants, then*

$$c_0\phi_0 + c_1\phi_1 \tag{4.35}$$

*is also a solution.*

As in Example 4.1, the proof is by the sum- and constant-rules of differential calculus. Since the 0-function is a solution of (4.34), another way to state the conclusion of the theorem is that the set of solutions of (4.34) is a **vector-space**.

Suppose one does know two solutions as in the theorem. Then a solution satisfying (4.33) can be found by solving

$$\begin{cases} y_0 = c_0\phi_0(t_0) + c_1\phi_1(t_0), \\ y_0' = c_0\phi_0'(t_0) + c_1\phi_1'(t_0). \end{cases} \tag{4.36}$$

This system has a solution if the determinant

$$\begin{vmatrix} \phi_0(t_0) & \phi_1(t_0) \\ \phi_0'(t_0) & \phi_1'(t_0) \end{vmatrix} \tag{4.37}$$

is not 0. This determinant is called the **Wronskian** of $\phi_0$ and $\phi_1$ at $t_0$ and can be denoted

$$W(\phi_0, \phi_1)(t_0). \tag{4.38}$$

Suppose this is *not* 0; then:

- for the equation (4.34) with initial conditions (4.33) there is a solution of the form (4.35);
- the functions $\phi_0$ and $\phi_1$ compose a **fundamental set of solutions** to (4.34);
- equation (4.35) can be called a **general solution** to (4.34);
- the space of solutions to (4.34) is **2-dimensional**.

**4.4 Example.** Suppose the characteristic equation of (4.4) has distinct real roots $r_0$ and $r_1$. Let $\phi_i(t) = e^{r_i t}$. Then

$$W(\phi_0, \phi_1)(t) = \begin{vmatrix} e^{r_0 t} & e^{r_1 t} \\ r_0 e^{r_0 t} & r_1 e^{r_1 t} \end{vmatrix} = (r_1 - r_0)e^{(r_0 + r_1)t} \neq 0, \tag{4.39}$$

so $\phi_0$ and $\phi_1$ compose a pair of fundamental solutions of (4.4).

**4.5 Example.** The equation

$$2t^2y'' + 3ty' - 1 = 0 \tag{4.40}$$

can be written

$$2y'' + 3t^{-1}y' - t^{-2} = 0, \tag{4.41}$$

so by the claim at the beginning of this sub-section, the equation has solutions on $(0, \infty)$. One can check that two such solutions are $t^{1/2}$ and $t^{-1}$. These are fundamental solutions since

$$W(t^{1/2}, t^{-1}) = \begin{vmatrix} t^{1/2} & t^{-1} \\ t^{-1/2}/2 & -t^{-2} \end{vmatrix} = \frac{-3t^{-3/2}}{2} \neq 0 \tag{4.42}$$

when $t > 0$.

**4.6 Example.** Suppose $\phi_i$ are solutions of (4.34) satisfying the initial conditions

$$\phi_0(t_0) = 1, \quad \phi_0'(t_0) = 0, \tag{4.43}$$
$$\phi_1(t_0) = 0, \quad \phi_1'(t_0) = 1. \tag{4.44}$$

Then the $\phi_i$ compose a fundamental pair of solutions, since their Wronskian is 1.

The last example shows that, by the claim at the beginning of the sub-section, *fundamental solutions always exist.* More than one pair exists for a given equation:

**4.7 Example.** Equation (4.5) has fundamental solutions $e^t$ and $e^{-t}$. If we solve (4.5) for the fundamental solutions in the last example, we get $(e^t + e^{-t})/2$ and $(e^t - e^{-t})/2$, that is, $\cosh t$ and $\sinh t$. So the general solution can be written

$$k_0 \cosh t + k_1 \sinh t. \tag{4.45}$$

Thus, (4.5) has two neat pairs of fundamental solutions, namely, $\{e^t, e^{-t}\}$ and $\{\cosh, \sinh\}$.

If $W(\phi_0, \phi_1)(t_0) \neq 0$ for some $t_0$, then the *homogeneous* linear system

$$\begin{cases} c_0\phi_0(t_0) + c_1\phi_1(t_0) = 0, \\ c_0\phi_0'(t_0) + c_1\phi_1'(t_0) = 0. \end{cases} \tag{4.46}$$

has the unique solution $(0, 0)$. This means that the only solution to the equation

$$c_0\phi_0 + c_1\phi_1 = 0 \tag{4.47}$$

is $(0, 0)$. By definition then, the functions $\phi_0$ and $\phi_1$ are **linearly independent**. We have shown:

**4.8 Theorem.** *If the Wronskian of two functions is non-zero at one point, then the two functions are linearly independent.*

The converse is false: that is, there are linearly independent functions whose Wronskian is 0 at all points:

**4.9 Example.** On the interval $(-2, 2)$, let $\phi_0(t) = t^3$ and let $\phi_1(t) = \left|t^3\right| = t^2\,|t|$. Then $\phi_0'(t) = 3t^2$, while $\phi_1'(t) = 3t\,|t|$. Hence

$$W(\phi_0, \phi_1)(t) = \begin{vmatrix} t^3 & t^2\,|t| \\ 3t^2 & 3t\,|t| \end{vmatrix} = 0 \tag{4.48}$$

for all $t$. However, suppose $c_0\phi_0 + c_1\phi_1 = 0$ on $(-2, 2)$. Then at 1 we have $c_0 + c_1 = 0$, while at $-1$ we have $-c_0 + c_1 = 0$; solved simultaneously, these equations have the unique solution $(0, 0)$. Therefore $\phi_0$ and $\phi_1$ are linearly independent.

**4.10 Example.** If $r_0 \neq r_1$, then the functions $e^{r_0 t}$ and $e^{r_1 t}$ are linearly independent by Example 4.4.

If the $\phi_i$ are solutions to a second-order linear homogeneous differential equation, then examples like 4.9 are not possible:

**4.11 Theorem.** *Suppose $\phi_0$ and $\phi_1$ are defined on an open interval $I$ and are solutions of*

$$y'' + p(t)y' + q(t)y = 0. \tag{4.49}$$

*If their Wronskian is 0 at one point of $I$, then the functions are linearly dependent.*

*Proof.* Suppose $W(\phi_0, \phi_1)(t_0) = 0$. Then the system (4.46) has a non-zero solution $(c_0, c_1)$. Let $\psi$ be the function $c_0\phi_0 + c_1\phi_1$. Then, by (4.46), we have $\psi(t_0) = 0$ and $\psi'(t_0) = 0$. But $\psi$ is also a solution of (4.49), and there is only one solution meeting the initial conditions $y(t_0) = 0$ and $y'(t_0) = 0$, namely, $y = 0$. Hence $\psi = 0$, which means that the $\phi_i$ are linearly dependent. □

We now know that the following are equivalent conditions on a pair $\{\phi_0, \phi_1\}$ of solutions of (4.49):

(1) $W(\phi_0, \phi_1)(t_0) \neq 0$ for some $t_0$.

(2) $\{\phi_0, \phi_1\}$ is a fundamental set of solutions of (4.49).

(3) The $\phi_i$ are linearly independent.

(4) $W(\phi_0, \phi_1)(t_0) \neq 0$ for all $t_0$ (in the domain of the $\phi_i$).

In particular, we can say something about $W(\phi_0, \phi_1)$ without knowing the $\phi_i$: it is 0 either everywhere or nowhere. We can say more:

**4.12 Theorem** (Abel)**.** *If $\phi_0$ and $\phi_1$ are solutions of* (4.49)*, then*

$$W(\phi_0, \phi_1)(t) = c \exp\left(-\int^t p\right), \tag{4.50}$$

*where $c$ is constant (that is, $c$ is independent of $t$, or $\mathrm{d}\,c/\,\mathrm{d}\,t = 0$; the value of $c$ does depend on $\phi_0$ and $\phi_1$).*

*Proof.* Write the Wronskian here as $W$. We have

$$W = \phi_0\phi_1' - \phi_1\phi_0', \tag{4.51}$$

$$W' = \phi_0\phi_1'' - \phi_1\phi_0''. \tag{4.52}$$

From (4.49), we also have

$$\phi_1\phi_0'' + p\phi_1\phi_0' + q\phi_1\phi_0 = 0, \tag{4.53}$$
$$\phi_0\phi_1'' + p\phi_0\phi_1' + q\phi_0\phi_1 = 0, \tag{4.54}$$

which by subtraction yields

$$W' + pW = 0. \tag{4.55}$$

This is a first-order differential equation we can solve, and its solution is (4.50). □

## 4.3 Homogeneous equations with constant coefficients, II

By observation, the equation

$$y'' + y = 0 \tag{4.56}$$

has a fundamental set $\{\cos, \sin\}$ of solutions. The characteristic equation

$$r^2 + 1 = 0 \tag{4.57}$$

has roots $\pm i$. Suppose the equation

$$r^2 + ar + b = 0 \tag{4.58}$$

has roots $\lambda \pm i\mu$, where $\lambda$ and $\mu$ are real, and $\mu \neq 0$ (so that the two roots are distinct). One may guess that a fundamental set of solutions of

$$y'' + ay' + by = 0 \tag{4.59}$$

is $\{e^{\lambda t}\cos\mu t, e^{\lambda t}\sin\mu t\}$. The guess is correct, as we can verify: For example, let

$$\phi(t) = e^{\lambda t}\cos\mu t; \tag{4.60}$$

then

$$\phi'(t) = \lambda e^{\lambda t}\cos\mu t - \mu e^{\lambda t}\sin\mu t, \tag{4.61}$$
$$\phi''(t) = \lambda^2 e^{\lambda t}\cos\mu t - 2\lambda\mu e^{\lambda t}\sin\mu t - \mu^2 e^{\lambda t}\cos\mu t, \tag{4.62}$$

and therefore

$$(\phi'' + a\phi' + b\phi)(t) = (\lambda^2 - \mu^2 + a\lambda + b)e^{\lambda t}\cos\mu t - \mu(2\lambda + a)e^{\lambda t}\sin\mu t. \tag{4.63}$$

By assumption, we have

$$r^2 + ar + b = (r - \lambda - i\mu)(r - \lambda + i\mu) \tag{4.64}$$
$$= (r - \lambda)^2 - (i\mu)^2 \tag{4.65}$$
$$= r^2 - 2r\lambda + \lambda^2 + \mu^2, \tag{4.66}$$

whence

$$a = -2\lambda, \tag{4.67}$$
$$b = \lambda^2 + \mu^2 \tag{4.68}$$

and therefore

$$2\lambda + a = 0, \tag{4.69}$$

$$\lambda^2 - \mu^2 + a\lambda + b = \lambda^2 - \mu^2 - 2\lambda^2 + \lambda^2 + \mu^2 = 0, \tag{4.70}$$

$$\phi'' + a\phi' + b\phi = 0. \tag{4.71}$$

Likewise if $\phi(t) = e^{\lambda t}\sin\mu t$. Therefore $e^{\lambda t}\cos\mu t$ and $e^{\lambda t}\sin\mu t$ are solutions of (4.59). They comprise a fundamental set since their Wronskian is non-zero:

$$\begin{vmatrix} e^{\lambda t}\cos\mu t & e^{\lambda t}\sin\mu t \\ e^{\lambda t}(\lambda\cos\mu t - \mu\sin\mu t) & e^{\lambda t}(\lambda\sin\mu t + \mu\cos\mu t) \end{vmatrix} =$$

$$e^{2\lambda t}\left(\lambda\begin{vmatrix}\cos\mu t & \sin\mu t \\ \cos\mu t & \sin\mu t\end{vmatrix} + \mu\begin{vmatrix}\cos\mu t & \sin\mu t \\ -\sin\mu t & \cos\mu t)\end{vmatrix}\right) = \mu e^{2\lambda t}. \tag{4.72}$$

Why does this approach work? We might expect complex exponentiation to be defined so that (4.59) has solutions $e^{(\lambda\pm i\mu)t}$. In particular, if $\phi(t) = e^{it}$, then we should have

$$\phi'(t) = i\phi(t), \tag{4.73}$$

$$\phi''(t) = -\phi(t), \tag{4.74}$$

so that $\phi$ solves the initial-value problem

$$y'' + y = 0, \quad y(0) = 1, \quad y'(0) = i. \tag{4.75}$$

But this problem should have a unique solution $c_0\cos + c_1\sin$, where—since $(c_0\cos + c_1\sin)' = -c_0\sin + c_1\cos$—we have

$$1 = c_0\cos 0 + c_1\sin 0 = c_0, \tag{4.76}$$

$$i = -c_0\sin 0 + c_1\cos 0 = c_1. \tag{4.77}$$

Therefore we should define

$$e^{it} = \cos t + i\sin t; \tag{4.78}$$

this is known as **Euler's formula**. From it we have

$$e^{-it} = \cos t - i\sin t, \tag{4.79}$$

whence

$$e^{(\lambda\pm i\mu)t} = e^{\lambda t}e^{\pm i\mu t} = e^{\lambda t}(\cos\mu t \pm i\sin\mu t). \tag{4.80}$$

Hence, by Euler's formula, $e^{(\lambda\pm i\mu)t}$ *are* solutions of (4.59). Moreover, the solutions we found originally are given by

$$e^{\lambda t}\cos\mu t = \frac{1}{2}(e^{(\lambda+i\mu)t} + e^{(\lambda-i\mu)t}), \tag{4.81}$$

$$e^{\lambda t}\sin\mu t = \frac{1}{2i}(e^{(\lambda+i\mu)t} - e^{(\lambda-i\mu)t}). \tag{4.82}$$

$$\tag{4.83}$$

Thus $\{e^{(\lambda\pm i\mu)t}\}$ is a fundamental set of solutions of (4.59).

Note in particular

$$\cos t = \frac{1}{2}(e^{it} + e^{it}), \tag{4.84}$$

$$\sin t = \frac{1}{2i}(e^{it} - e^{it}). \tag{4.85}$$

**4.13 Example.** The equation

$$y'' - y' + y = 0 \tag{4.86}$$

has characteristic equation

$$r^2 - r + 1 = 0, \tag{4.87}$$

whence

$$r = \frac{1 \pm i\sqrt{3}}{2}, \tag{4.88}$$

so the general solution is

$$e^{t/2}\left(c_0 \cos\frac{t\sqrt{3}}{2} + c_1 \sin\frac{t\sqrt{3}}{2}\right). \tag{4.89}$$

**4.14 Remark.** The product

$$e^{\lambda t}(c_0 \cos \mu t + i \sin \mu t) \tag{4.90}$$

can also be written

$$Ae^{\lambda t} \sin(\mu t + \theta) \tag{4.91}$$

for some $A$ and $\theta$. One can verify this by angle-addition formulas. Alternatively, $A\sin(t+\theta)$ is a solution of (4.56), so it is $c_0 \cos t + c_1 \sin t$ for some $c_0$ and $c_1$; moreover,

$$A \sin\theta = c_0, \tag{4.92}$$

$$A \cos\theta = c_1. \tag{4.93}$$

From this, $A^2 = c_0^2 + c_1^2$, and then $\theta$ is uniquely determined in the interval $[0, 2\pi)$.

## 4.4 Homogeneous equations with constant coefficients, III

Suppose the characteristic equation (4.58) has a double root—which must therefore be $-a/2$. Then (4.59) has the solution $e^{-at/2}$. Call this $\phi(t)$. Then $t\phi(t)$ is also a solution. Indeed, we have

$$\frac{\mathrm{d}}{\mathrm{d}\,t}(t\phi(t)) = \phi(t) - \frac{a}{2}t\phi(t), \tag{4.94}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}\,t^2}(t\phi(t)) = -a\phi(t) + \frac{a^2}{4}t\phi(t) \tag{4.95}$$

and hence

$$\frac{\mathrm{d}^2}{\mathrm{d}\,t^2}(t\phi(t)) + a\frac{\mathrm{d}}{\mathrm{d}\,t}(t\phi(t)) + bt\phi(t) = \left(\frac{a^2}{4} - \frac{a^2}{2} + b\right)t\phi(t) = 0 \tag{4.96}$$

since $4b = a^2$. Moreover, $\phi(t)$ and $t\phi(t)$ are linearly independent, since

$$\begin{vmatrix} \phi(t) & t\phi(t) \\ \phi'(t) & \phi(t) + t\phi'(t) \end{vmatrix} = \begin{vmatrix} \phi(t) & t\phi(t) \\ 0 & \phi(t) \end{vmatrix} + \begin{vmatrix} \phi(t) & t\phi(t) \\ \phi'(t) & t\phi'(t) \end{vmatrix} = \phi(t)^2. \tag{4.97}$$

Thus $\{e^{-at/2}, te^{-at/2}\}$ is a fundamental set of solutions of (4.59) when the characteristic equation has a double root.

**4.15 Example.** The equation

$$y'' - 2y + y = 0 \tag{4.98}$$

has characteristic equation

$$0 = r^2 - 2r + 1 = (r-1)^2, \tag{4.99}$$

so $\{e^t, te^t\}$ is a fundamental set of solutions. Let $\phi(t) = c_0 e^t + c_1 te^t$, so that $\phi'(t) = (c_0 + c_1)e^t + c_1 te^t$. The initial conditions

$$y(0) = 1 \quad \text{and} \quad y'(0) = 0 \tag{4.100}$$

give us

$$\begin{cases} 1 = c_0, \\ 0 = c_0 + c_1, \end{cases} \tag{4.101}$$

so the solution is $e^t - te^t$, that is, $(1-t)e^t$.

## 4.5 Reduction of order

The solution $te^{-at/2}$ in the previous subsection can be derived from the solution $e^{-at/2}$ by **reduction of order**, that is, by reducing the original second-order equation (4.59) to a first-order equation. The general procedure is as follows.

Suppose $\phi$ is a solution of of (4.34). We try to find a function $v$ so that $v\phi$ is also a solution. We have

$$(v\phi)' = v\phi' + v'\phi, \tag{4.102}$$

$$(v\phi)'' = v\phi'' + 2v'\phi' + v''\phi, \tag{4.103}$$

whence

$$(v\phi)'' + p(v\phi)' + qv\phi = 2v'\phi' + v''\phi + pv'\phi. \tag{4.104}$$

Thus $v$ should be a solution of the equation

$$2v'\phi' + v''\phi + pv'\phi = 0, \tag{4.105}$$

which means $v'$ should be a solution of

$$\phi(v')' + (2\phi' + p\phi)v' = 0, \tag{4.106}$$

a *first-order* equation—in fact, linear-homogeneous (hence separable).

**4.16 Example.** If our equation is (4.98), with solution $e^t$, then (4.106) becomes

$$e^t v''(t) = 0, \tag{4.107}$$

so $v'$ is constant, and we may let $v(t) = t$.

**4.17 Example.** The equation

$$2t^2y'' + 3ty' - y = 0 \tag{4.108}$$

has the solution $1/t$. We want another solution, $v(t)/t$. We have

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\left(\frac{v(t)}{t}\right) = \frac{tv'(t) - v(t)}{t^2} = \frac{v'(t)}{t} - \frac{v(t)}{t^2}, \tag{4.109}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}\,t^2}\left(\frac{v(t)}{t}\right) = \frac{tv''(t) - v'(t)}{t^2} - \frac{t^2v'(t) - 2tv(t)}{t^4} \tag{4.110}$$

$$= \frac{t^2v''(t) - 2tv'(t) + 2v(t)}{t^3}, \tag{4.111}$$

so replacing $y$ with $v(t)/t$ in (4.108) gives

$$2(t^2v''(t) - 2tv'(t) + 2v(t)) + 3(tv'(t) - v(t)) - v(t) = 0, \tag{4.112}$$

$$2t^2v''(t) - tv'(t) = 0, \tag{4.113}$$

$$\frac{v''(t)}{v'(t)} = \frac{1}{2t}, \tag{4.114}$$

$$\ln v'(t) = \frac{1}{2}\ln t + C; \tag{4.115}$$

letting $C$ be 0, we get

$$v'(t) = t^{1/2}, \tag{4.116}$$

$$v = \int t^{1/2}\,\mathrm{d}\,t = \frac{2}{3}t^{3/2} + C, \tag{4.117}$$

where again we may let $C$ be 0.

## 4.6 Inhomogeneous equations

We know that the set of solutions of (4.34) is a two-dimensional vector-space $V$. If $\psi$ is a solution of

$$y'' + p(t)\cdot y' + q(t)\cdot y = g(t), \tag{4.118}$$

then the set of solutions is $V + \psi$, the **translation** of $V$ by $\psi$. This means two things:

- If $\psi_0$ and $\psi_1$ solve (4.118), then $\psi_0 - \psi_1$ solves (4.34).
- If $\psi$ solves (4.118), and $\phi$ solves (4.34), then $\psi + \phi$ solves (4.118).

So if we know the general solution to (4.34), then we know every solution to (4.118), *provided* we know just one solution. To find one solution, we have two methods:

**4.18 Example.** To solve

$$y'' + 3y' - 4y = e^{2t}, \tag{4.119}$$

we note first that the general solution of the corresponding homogeneous equation

$$y'' + 3y' - 4y = e^{2t} \tag{4.120}$$

is $c_0 e^t + c_1 e^{-4t}$. We guess that the inhomogeneous equation has solution $Ae^{2t}$, where $A$ is an **undetermined coefficient**. If $\phi(t) = Ae^{2t}$, then $\phi' = 2\phi$ and $\phi'' = 4\phi$, so we want

$$(\phi'' + 3\phi' - 4\phi)(t) = e^{2t}, \tag{4.121}$$

$$(4 + 3 \cdot 2 - 4)Ae^{2t} = e^{2t}, \tag{4.122}$$

$$6A = 1. \tag{4.123}$$

So the inhomogeneous equation has general solution

$$c_0 e^t + c_1 e^{-4t} + \frac{1}{6} e^{2t}. \tag{4.124}$$

If $\phi_i$ solves $y'' + p \cdot y' + q \cdot y = g_i$, then $\phi_0 + \phi_1$ solves

$$y'' + p \cdot y' + q \cdot y = g_0 + g_1. \tag{4.125}$$

So the method of undetermined coefficients has a broader application:

**4.19 Example.** To find a particular solution of

$$y'' + 3y' - 4y = e^{2t} - e^{6t}, \tag{4.126}$$

it is enough to take $e^{2t}/6$ from the previous example and add it to a solution of $y'' + 3y' - 4y = -e^{6t}$, such as $-e^{6t}/50$.

**4.20 Example.** To solve

$$y'' + 3y' - 4y = 2 \sin t, \tag{4.127}$$

remember that $2i \sin t = e^{it} - e^{-it}$; so we can look for a particular solution of the form

$$Ce^{it} + De^{-it}; \tag{4.128}$$

but since also $2 \cos t = e^{it} + e^{-it}$, we can also look for a solution

$$A \cos t + B \sin t. \tag{4.129}$$

## 4.7 Variation of parameters

[Not typed up yet.]

# 5 Laplace transform

## 5.1 Basics

Suppose $f$ is a function defined on $[0, \infty)$. The function $f$ (if it exists) defined on an interval $[p, \infty)$ by

$$F(s) = \int_0^\infty e^{-st} f(t) \, \mathrm{d}\, t \tag{5.1}$$

is called the **Laplace transform** of $f$ and is denoted

$$\mathcal{L}\{f\}. \tag{5.2}$$

By convention here, $f$ is treated as $f(t)$, and $F$ as $F(s)$.
**5.1 Example.** By integration by parts, we have

$$\begin{aligned}
\mathcal{L}\{t\} &= \int_0^\infty te^{-st}\,\mathrm{d}\,t && (5.3)\\
&= \lim_{x\to\infty}\int_0^x te^{-st}\,\mathrm{d}\,t && (5.4)\\
&= \lim_{x\to\infty}\left(\left[\frac{te^{-st}}{-s}\right]_0^x + \frac{1}{s}\int_0^x e^{-st}\,\mathrm{d}\,t\right) && (5.5)\\
&= \lim_{x\to\infty}\left[\frac{e^{-st}}{-s^2}\right]_0^x && (5.6)\\
&= \frac{1}{s^2}. && (5.7)
\end{aligned}$$

One important property of $\mathcal{L}\{\}$ is easily proved:
**5.2 Theorem.** *$\mathcal{L}\{\}$ is linear, that is,*

$$\mathcal{L}\{af+bg\} = a\mathcal{L}\{f\} + b\mathcal{L}\{g\}, \tag{5.8}$$

*where $f$ and $g$ are functions, and $a$ and $b$ are constants.*

To use $\mathcal{L}\{\}$ in solving differential equations, we can relate $\mathcal{L}\{f\}$ and $\mathcal{L}\{f'\}$. First we identify a condition on $f$ guaranteeing that $\mathcal{L}\{f\}$ exists:
**5.3 Theorem.** *If $f$ is defined on $[0,\infty)$ and is integrable on each interval $[0,t]$, and if $\lim_{t\to\infty} e^{-pt}f(t) = 0$ for some (positive) $p$, then $\mathcal{L}\{f\}$ exists.*

*Proof.* By hypothesis, there is $M$ such that, if $M \leqslant t$, then $|e^{-pt}f(t)| \leqslant 1$. Suppose $p < s$. Then $|e^{-st}f(t)| \leqslant e^{(p-s)t}\,|e^{-pt}f(t)| \leqslant e^{(p-s)t}$ when $M \leqslant t$. Since $\int_0^\infty e^{(p-s)t}$ exists, $\mathcal{L}\{f\}$ must exist. □

If $f$ meets the condition that $\lim_{t\to\infty} e^{-pt}f(t) = 0$ for some (positive) $p$, then $f$ may be said to have **exponential order**.
**5.4 Theorem.** *If $f$ is continuously differentiable and has exponential order, then $\mathcal{L}\{f'\}$ exists and is given by*

$$\mathcal{L}\{f'\} = s\cdot\mathcal{L}\{f\} - f(0). \tag{5.9}$$

*Proof.* Using integration by parts, we have

$$\begin{aligned}
\mathcal{L}\{f'\} &= \int_0^\infty e^{-st}f'(t)\,\mathrm{d}\,t && (5.10)\\
&= \left[e^{-st}f(t)\right]_0^\infty + s\int_0^\infty e^{-st}f(t)\,\mathrm{d}\,t && (5.11)\\
&= \left[e^{-st}f(t)\right]_0^\infty + s\mathcal{L}\{f(t)\}. && (5.12)
\end{aligned}$$

Since $f$ has exponential order, we must have

$$\lim_{x\to\infty} e^{-sx} f(x) = 0 \tag{5.13}$$

if $s$ is large enough. The claim now follows. □

In fact the last theorem is true if $f'$ is merely 'piecewise' continuous. A third fact we shall not prove:

**5.5 Theorem.** *If* $\mathcal{L}\{f\} = \mathcal{L}\{g\}$*, then* $f = g$.

We can now solve some initial-value problems:

**5.6 Example.** To solve

$$y' - 2y = e^{4t}, \quad y(0) = 3, \tag{5.14}$$

suppose $\phi$ *is* a solution. Then

$$\mathcal{L}\{\phi' - 2\phi\} = \mathcal{L}\left\{e^{4t}\right\}, \tag{5.15}$$

$$\mathcal{L}\{\phi'\} - 2\cdot\mathcal{L}\{\phi\} = \int_0^\infty e^{-st} e^{4t}\,\mathrm{d}\,t, \tag{5.16}$$

$$s\cdot\mathcal{L}\{\phi\} - \phi(0) - 2\cdot\mathcal{L}\{\phi\} = \frac{1}{s-4}, \tag{5.17}$$

$$(s-2)\cdot\mathcal{L}\{\phi\} - 3 = \frac{1}{s-4}, \tag{5.18}$$

$$(s-2)\cdot\mathcal{L}\{\phi\} = 3 + \frac{1}{s-4} = \frac{3s-11}{s-4}, \tag{5.19}$$

$$\mathcal{L}\{\phi\} = \frac{3s-11}{(s-2)(s-4)}. \tag{5.20}$$

Now decompose the right side into 'partial fractions':

$$\frac{3s-11}{(s-2)(s-4)} = \frac{A}{s-2} + \frac{B}{s-4}, \tag{5.21}$$

where

$$3s - 11 = A(s-4) + B(s-2) = (A+B)s - 4A - 2B, \tag{5.22}$$

whence

$$\begin{cases} A + B = 3, \\ 4A + 2B = 11, \end{cases} \tag{5.23}$$

so that

$$A = \frac{\begin{vmatrix} 3 & 1 \\ 11 & 2 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 4 & 2 \end{vmatrix}} = \frac{6-11}{2-4} = \frac{5}{2}, \quad B = 3 - \frac{5}{2} = \frac{1}{2}. \tag{5.24}$$

We now have

$$\mathcal{L}\{\phi\} = \frac{5}{2}\left(\frac{1}{s-2}\right) + \frac{1}{2}\left(\frac{1}{s-4}\right). \tag{5.25}$$

Our calculation of $\mathcal{L}\left\{e^{4t}\right\}$ can be adapted to show

$$\mathcal{L}\left\{e^{at}\right\} = \frac{1}{s-a}. \tag{5.26}$$

So we have

$$\mathcal{L}\{\phi\} = \frac{5}{2}\mathcal{L}\left\{e^{2t}\right\} + \frac{1}{2}\mathcal{L}\left\{e^{4t}\right\} \tag{5.27}$$

$$= \mathcal{L}\left\{\frac{5}{2}e^{2t} + \frac{1}{2}e^{4t}\right\}, \tag{5.28}$$

and therefore

$$\phi = \frac{5}{2}e^{2t} + \frac{1}{2}e^{4t}. \tag{5.29}$$

To solve higher-order differential equations, we can use a generalization of Theorem 5.4:

**5.7 Theorem.** *If $f^{(n-1)}$ is continuously differentiable and has exponential order, then $\mathcal{L}\left\{f^{(n)}\right\}$ exists and is given by*

$$\mathcal{L}\left\{f^{(n)}\right\} = s^n \cdot \mathcal{L}\{f\} - \sum_{k=0}^{n-1} s^k f^{(n-1-k)}(0). \tag{5.30}$$

*Proof.* Induction. □

**5.8 Example.** To solve

$$y'' + y = e^{-2t}\sin t, \quad y(0) = 0, \quad y'(0) = 0, \tag{5.31}$$

apply $\mathcal{L}\{\}$ to the differential equation, and use the initial values:

$$\mathcal{L}\{y''\} + \mathcal{L}\{y\} = \mathcal{L}\left\{e^{-2t}\sin t\right\}, \tag{5.32}$$

$$s^2 \cdot \mathcal{L}\{y\} - s \cdot y(0) - y'(0) + \mathcal{L}\{y\} = \mathcal{L}\left\{e^{-2t}\sin t\right\}, \tag{5.33}$$

$$(s^2+1) \cdot Y = \mathcal{L}\left\{e^{-2t}\sin t\right\}, \tag{5.34}$$

where $Y = \mathcal{L}\{y\}$.

To continue with the solution, we need more facts about Laplace transforms:

**5.9 Theorem.** *If $\mathcal{L}\{f(t)\} = F(s)$, then $F(s+a) = \mathcal{L}\{e^{-at}f(t)\}$.*

So $\mathcal{L}\left\{e^{-2t}\sin t\right\} = F(s+2)$, where $F(s) = \mathcal{L}\{\sin t\}$. Now,

$$\mathcal{L}\{\sin\} = \int_0^\infty e^{-st}\sin t \,\mathrm{d}\, t \tag{5.35}$$

$$= \left[-e^{-st}\cos t\right]_0^\infty - s\int_0^\infty e^{-st}\cos t \,\mathrm{d}\, t \tag{5.36}$$

$$= 1 - s \cdot \mathcal{L}\{\cos\}, \tag{5.37}$$

and similarly

$$\mathcal{L}\{\cos\} = s \cdot \mathcal{L}\{\sin\} \tag{5.38}$$

$$= s - s^2 \cdot \mathcal{L}\{\cos\}, \tag{5.39}$$

so that

$$\mathcal{L}\{\cos\} = \frac{s}{s^2+1}, \quad \mathcal{L}\{\sin\} = \frac{1}{s^2+1}. \tag{5.40}$$

Alternatively, one can use that

$$\mathcal{L}\{\cos\} + i \cdot \mathcal{L}\{\sin\} = \int_0^\infty e^{-st}(\cos t + i \sin t)\,\mathrm{d}\,t \tag{5.41}$$

$$= \int_0^\infty e^{-st} e^{it}\,\mathrm{d}\,t = \left[\frac{1}{-s+i} e^{(-s+i)t}\right]_0^\infty \tag{5.42}$$

$$= \frac{1}{s-i} \tag{5.43}$$

$$= \frac{s+i}{s^2+1}. \tag{5.44}$$

Continuing with Example 5.8, we have

$$(s^2+1) \cdot Y = \frac{1}{(s+2)^2+1}, \tag{5.45}$$

$$Y = \frac{1}{(s^2+1)(s^2+4s+5)}, \tag{5.46}$$

$$Y = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4s+5}, \tag{5.47}$$

where one should calculate that

$$A = -\frac{1}{8}, \quad B = \frac{1}{8}, \quad C = \frac{1}{8}, \quad D = \frac{3}{8}. \tag{5.48}$$

In any case, we now have

$$Y = A\frac{s}{s^2+1} + B\frac{1}{s^2+1} + C\frac{s}{(s+2)^2+1} + D\frac{1}{(s+2)^2+1} \tag{5.49}$$

$$= A\mathcal{L}\{\cos\} + B\mathcal{L}\{\sin\} + C\mathcal{L}\left\{e^{-2t}\cos t\right\} + D\mathcal{L}\left\{e^{-2t}\sin t\right\}, \tag{5.50}$$

so that

$$y = A\cos t + B\sin t + Ce^{-2t}\cos t + De^{-2t}\sin t. \tag{5.51}$$

In the example, if $\sin t$ had been replaced with $\sin 5t$, we could have used the following:

**5.10 Theorem.** *If $\mathcal{L}\{f\} = F$, then $\mathcal{L}\{f(at)\} = F(s/a)/a$.*

*Proof.* By the substitution

$$u = at, \quad \mathrm{d}\,u = a\,\mathrm{d}\,t, \tag{5.52}$$

we have

$$\mathcal{L}\{f(at)\} = \int_0^\infty e^{-st} f(at)\,\mathrm{d}\,t \tag{5.53}$$

$$= \frac{1}{a}\int_0^\infty e^{-su/a} f(u)\,\mathrm{d}\,u \tag{5.54}$$

$$= \frac{1}{a} \cdot F\left(\frac{s}{a}\right), \tag{5.55}$$

where $F(s) = \mathcal{L}\{f\}$. □

Hence for example

$$\mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}, \quad \mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}. \tag{5.56}$$

## 5.2 Springs

Suppose an object is subject to the force of a spring (and gravity). Let $u(t)$ be the displacement of the object from its resting position. Then the force on the spring is $mu''(t)$. The force may have three components:

(1) $-ku(t)$, by Hooke's Law;

(2) $-\gamma u'(t)$, due to friction (**damping**);

(3) some other force $F(t)$.

Thus arises the 2nd-order linear differential equation

$$mu'' + \gamma u' + ku = F(t), \tag{5.57}$$

where the coefficients $m$, $\gamma$ and $k$ are non-negative constants. Laplace transforms will be particularly useful in case $F$ has discontinuities.

## 5.3 Discontinuities

**5.11 Example.** To solve

$$y'' + 2y' + 5y = F(t), \quad y(0) = 1, \quad y'(0) = 0, \tag{5.58}$$

where

$$F(t) = \begin{cases} 1, & \text{if } 0 \leqslant t < \pi; \\ 0, & \text{if } \pi \leqslant t, \end{cases} \tag{5.59}$$

note that

$$\mathcal{L}\{F\} = \int_0^\infty e^{-st} F(t)\,\mathrm{d}\,t \tag{5.60}$$

$$= \int_0^\pi e^{-st}\,\mathrm{d}\,t \tag{5.61}$$

$$= \left[-\frac{e^{-st}}{s}\right]_0^\pi \tag{5.62}$$

$$= \frac{1 - e^{s\pi}}{s}. \tag{5.63}$$

So we have

$$\mathcal{L}\{y''\} + 2\mathcal{L}\{y'\} + 5\mathcal{L}\{y\} = \frac{1 - e^{s\pi}}{s}, \tag{5.64}$$

$$s^2 Y - s + 2sY - 2 + 5Y = \frac{1 - e^{s\pi}}{s}, \tag{5.65}$$

$$(s^2 + 2s + 5)Y = \frac{1 - e^{s\pi}}{s} + s + 2, \tag{5.66}$$

$$(s^2 + 2s + 5)Y = \frac{s^2 + 2s + 1}{s} - \frac{e^{s\pi}}{s}, \tag{5.67}$$

$$Y = \frac{A}{s} + \frac{Bs + C}{s^2 + 2s + 5} - e^{s\pi}\left(\frac{D}{s} + \frac{Es + F}{s^2 + 2s + 5}\right), \tag{5.68}$$

$$Y = \frac{A}{s} + \frac{Bs + C}{(s+1)^2 + 4} - e^{s\pi}\left(\frac{D}{s} + \frac{Es + F}{(s+1)^2 + 4}\right). \tag{5.69}$$

To complete the solution, we need the following:...

# 6 Systems of differential equations

We want to solve systems of the form

$$\begin{cases} x_1' = a_{1\,1}x_1 + a_{1\,2}x_2 + \cdots + a_{1\,n}x_n \\ x_2' = a_{2\,1}x_1 + a_{2\,2}x_2 + \cdots + a_{2\,n}x_n \\ \cdots \\ x_n' = a_{n\,1}x_1 + a_{n\,2}x_2 + \cdots + a_{n\,n}x_n \end{cases} \tag{6.1}$$

We can write this as the matrix equation

$$\boldsymbol{x}' = A\boldsymbol{x}, \tag{6.2}$$

where

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} a_{1\,1} & \dots & a_{1\,n} \\ \vdots & & \vdots \\ a_{n\,1} & \dots & a_{n\,n} \end{bmatrix}. \tag{6.3}$$

In case $n = 1$, we have $x' = ax$, with general solution $ce^{at}$; that is, all solutions are linear combinations of $e^{at}$. In the general case, we look for solutions like $e^{at}$. They should be vectors—say $\boldsymbol{\xi}e^{rt}$, that is,

$$\begin{bmatrix} \xi_1 e^{rt} \\ \vdots \\ \xi_n e^{rt} \end{bmatrix}. \tag{6.4}$$

Plugging into (6.2) gives

$$r\boldsymbol{\xi}e^{rt} = A\boldsymbol{\xi}e^{rt}, \tag{6.5}$$

$$r\boldsymbol{\xi} = A\boldsymbol{\xi}, \tag{6.6}$$

$$rI\boldsymbol{\xi} = A\boldsymbol{\xi}, \tag{6.7}$$

$$(rI - A)\boldsymbol{\xi} = \mathbf{0}, \tag{6.8}$$

where

$$I = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}. \tag{6.9}$$

Here $r$ is an **eigenvalue** of $A$, and $\boldsymbol{\xi}$ is a corresponding **eigenvector**. This means $\boldsymbol{\xi} \neq \mathbf{0}$, and:

(1) Multiplying $\boldsymbol{\xi}$ by the matrix $A$ is the same as multiplying by a **scalar** (number);

(2) this scalar is $r$.

We find $r$ first by solving

$$\det(\lambda I - A) = 0, \tag{6.10}$$

the **characteristic equation** of $A$.

**6.1 Example.** Let $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$; the characteristic equation is

$$\begin{vmatrix} \lambda - 1 & 0 \\ 0 & \lambda + 1 \end{vmatrix} = 0, \tag{6.11}$$

$$\lambda^2 - 1 = 0, \tag{6.12}$$

$$\lambda = \pm 1. \tag{6.13}$$

That is, $A$ has eigenvalues $\pm 1$.

For the eigenvectors corresponding to 1, we solve:

$$(A - I)\begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{0}, \tag{6.14}$$

$$\begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{0}, \tag{6.15}$$

$$-2y = 0. \tag{6.16}$$

So the eigenvectors are the non-zero multiples of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

For the eigenvectors corresponding to $-1$, we solve:

$$(A + I)\begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{0}, \tag{6.17}$$

$$\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{0}, \tag{6.18}$$

$$2x = 0. \tag{6.19}$$

So the eigenvectors are the non-zero multiples of $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Hence the equation (6.2) has, in this case, the general solution

$$c_0 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t + c_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-t}. \tag{6.20}$$

Consider the direction-field...

**6.2 Example.** $A = \begin{bmatrix} 3 & -2 \\ 4 & -3 \end{bmatrix}$. Eigenvalues: $\pm 1$ again. Eigenvectors: $(1, 1)$ and $(1, 2)$.