Decidable and undecidable real closed rings

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1. Introduction

1.1. **Real closed fields**

Tarski has shown that the real field is decidable.

Proof:

We list a complete set of axioms, such as

(a) axioms for fields.

(b) the squares form the nonnegative elements of an order of the field.

(c) every univariate polynomial satisfies the intermediate value property with respect to the order. □

Completeness means that every sentence φ that is true in $\mathbb{R}$ can be formally derived from these axioms: A computer can now list all formal proofs and at some point φ or $\neg\varphi$ will appear on that list; at that time we have decided φ in $\mathbb{R}$.

Any field satisfying (a)-(c) is called real closed.

Examples: $\mathbb{R}$, $\mathbb{R}_{\mathrm{alg}}$, $\mathbb{R}((t^{\frac{1}{\infty}}))$

1.2. **Semi-algebraic terminology**

A semi-algebraic (s.a.) subset of $\mathbb{R}^n$ is a boolean combination of sets defined by polynomial inequalities $P(x_1, ..., x_n) \geq 0$, where P is a polynomial with coefficients in $\mathbb{R}$.
A s.a. function is a function $X \longrightarrow \mathbb{R}^k$ with s.a. graph (and so also $X \subseteq \mathbb{R}^n$ is s.a.).
Examples of s.a. functions: polynomials with integer coefficients, $\frac{1}{1+x^2}$, $\sqrt[3]{x}$, characteristic functions of s.a. sets, $\mathrm{dist}_S(\bar{x})$ (the distance function of some s.a. set $S \subseteq \mathbb{R}^n$).

Tarski: s.a. = definable in the field $\mathbb{R}$

If the polynomials defining a s.a. set/function have integer coefficients then we say $\mathbb{Z}$-s.a. instead of *s.a.* (in model theory, this called 0-definable).
S.a. functions behave neatly, e.g. for every s.a. function $\mathbb{R} \longrightarrow \mathbb{R}$ there are $-\infty = a_1 < ... < a_k = +\infty$ such that on each interval (a_i, a_{i+1}), f is continuous and strictly monotone or constant (o-minimality of $\mathbb{R}$).

1.3. **Generalisations**
Extensions of Tarski's theorem and its aftermath have been searched (and found). Essentially we want to find richer structures related to the real field that share some of the 'analysable' or 'tame' properties of $\mathbb{R}$ and of s.a. sets. Examples:

- o-minimal structures: add certain functions to $\mathbb{R}$ (van den Dries, Wilkie, ...)
- add a predicate for the set $2^{\mathbb{Z}} \subseteq \mathbb{R}$ (van den Dries)
- add a predicate "I am algebraic" to $\mathbb{R}$ (A. Robinson).

In this talk we are not looking for expansions of $\mathbb{R}$, but for rings that appear naturally in the real world.
Function algebras related to $\mathbb{R}$: $\mathbb{R}[x], \mathbb{R}(x), C^{\omega}(\mathbb{R}^n), C^{\infty}(\mathbb{R}^n), C(\mathbb{R}^n)$: all of those define $\mathbb{Z}$ (and $\mathbb{R}$), so are not tame.
On the other hand when studying s.a. sets it is natural to look at s.a. functions, in particular rings of continuous s.a. functions $X \longrightarrow \mathbb{R}$ and rings of germs of those, appear.

2. Definition of real closed rings

A real closed structure on a ring A is a family of functions

$$(f_A : A^n \longrightarrow A \mid n \in \mathbb{N}, f : \mathbb{R}^n \longrightarrow \mathbb{R} \text{ continuous and } \mathbb{Z}\text{-semi-algebraic}),$$

satisfying the following conditions:

S1: Given $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ and $g_1, \dots, g_n : \mathbb{R}^k \longrightarrow \mathbb{R}$ continuous and $\mathbb{Z}$-semi-algebraic, then

$$(f \circ (g_1, \dots, g_n))_A = f_A \circ ((g_1)_A, \dots, (g_n)_A)$$

S2: If $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is addition/multiplication of $\mathbb{R}$, then f_A is addition/multiplication of A. Hence $+_A = +$ and $\cdot_A = \cdot$. Similarly, $0_A = 0$ (in A), $1_A = 1$ (in A) and id_A is the identity $A \longrightarrow A$.

Examples: (a) Let $X \subseteq R^k$ be s.a. On

$$A = C_{s.a.}(X, \mathbb{R}) \text{ (defined as } \{f : X \longrightarrow \mathbb{R} \mid f \text{ continuous and s.a.}\})$$

we have a real closed str. given by

$$f_A(\alpha_1, \dots, \alpha_n) := f \circ (\alpha_1, \dots, \alpha_n) \text{ for } \alpha_1, \dots, \alpha_n \in C_{s.a.}(X, \mathbb{R}).$$

(b) On every real closed field A we have a real closed structure given by

$$f_A := \text{the function defined by the formula that defines } f.$$

2. Definition of real closed rings

Definition (N. Schwartz, 1981. Revamped in "Super Real Closed Rings")

A ring is called real closed if there exists a real closed structure on it.

So by the examples above, all real closed fields and all rings of (s.a.) real valued continuous functions are real closed.

To understand how the existence of a real closed structure on a ring can be used to obtain algebraic properties of A, lets do some teasers: Fix a real closed ring A and let $(f_A|...)$ be a real closed structure on A. Then

(i) A is reduced, i.e., whenever $a^n = 0$, then $a = 0$.

(ii) There is a partial order $\leq$ on A, given by $a \leq b \iff b - a$ is a square. With this partial order, A is a lattice ordered ring.

(iii) There is a natural map from the field of real algebraic numbers to A; if $1 \neq 0$ in A, this map is an embedding.

It should be noted that in the literature, there are various classes of rings that are also called real closed, but all these notions are different from each other (and different from the one we are using). The excellent properties of real closed ring on the next 4 slides are not valid for these classes.

3. **Fundamental theorem**

Let A be a real closed ring. Then

(a) There is a unique real closed structure on A, we refer to it by $(f_A|\ldots)$.

(b) Each f_A is definable in the ring A by some formula that defines the graph of f in $\mathbb{R}$ and this can be done independently of A. Such a formula can be chosen to be of the form $\exists \bar{u}\ P(\bar{x}, y, \bar{u}) = 0$ for some polynomial P with integer coefficients.

(c) Every ring homomorphism $h : A \longrightarrow B$ between real closed rings, preserves the real closed structures, i.e. $h(f_A(a)) = f_B(h(a))$. This is clear by the previous item.

(d) The class of real closed rings is first order axiomatizable, as follows easily from (b)

4. Properties of real closed rings

Let A be a real closed ring.

(a) If A is a field then A is a real closed field.

(b) If $I \subseteq A$ is a radical ideal, then also A/I is real closed.

(c) If $1 \in S \subseteq A$ is multiplicatively closed, then the localisation $S^{-1} \cdot A$ is real closed.

(d) Arbitrary limits and co-limits exist in the category of real closed rings.

(e) All rings of sections and all stalks of the Zariski sheaf on $\operatorname{Spec} A$ are real closed.

(f) Convex subrings of A are real closed.

(g) There is a largest real closed ring B that contains A as a convex subring.

(h) The Gelfand-Kolmogoroff theorem generalises to real closed ring. It says that for every real closed ring A and every convex subring B, the inclusion $B \hookrightarrow A$ induces a homeomorphism $\mathrm{Max}(A) \longrightarrow \mathrm{Max}(B)$.

$\curvearrowright$

Properties of real closed rings, continued

Example: The fibre product of two real closed rings in the category of rings is again real closed:

$$\begin{tikzcd} B & B \times_A C \arrow[l] \arrow[d] \\ A & C \arrow[l, "\psi"'] \arrow[from=1-1, to=2-1, "\varphi"'] \end{tikzcd} \qquad = \{(b, c) \mid \varphi(b) = \psi(c)\} \subseteq B \times C$$

5. The real closure

Every ring R has a unique real closure, up to a unique R-isomorphism. This means: there is a real closed ring $\rho(R)$ and a ring homomorphism $\rho_R : R \longrightarrow \rho(R)$ such that

$$\begin{tikzcd}
\rho(R) \arrow[r, dashed, "\exists!"] & A \\
R \arrow[u, "\rho_R"] \arrow[ur, "\forall"'] &
\end{tikzcd}$$

Examples:

- $\rho(\mathbb{C}) = 0$.
- The real closure of $\mathbb{Q}(\sqrt{2})$ is $\mathbb{R}_{\mathrm{alg}} \times \mathbb{R}_{\mathrm{alg}}$.
- $\rho(\mathbb{Z}[x_1, ..., x_n]) = \{f : \mathbb{R}^n \longrightarrow \mathbb{R} \mid f \text{ cont. and } \mathbb{Z}\text{-s.a.}\}$.

6. **Decidability, Case 0**

A very rough measure of how complicated a real closed ring A can be, is the prime spectrum $\mathrm{Spec}(A)$ of A. The easiest case is when $\mathrm{Spec}\, A$ is boolean:

(a) If $Spec(A)$ is a singleton, then A is a real closed field, therefore decidable by Tarski.

(b) If $Spec(A)$ is boolean, then A is von Neumann regular and a complete list of axioms of A is given by saying
   - A is a real closed ring,
   - A is von Neumann regular,
   - the Tarski invariants for the boolean algebra of idempotents of A.

   This follows from Feferman-Vaught. An example for such a ring is

   $$A = \{f : \mathbb{R}^n \longrightarrow \mathbb{R} \mid f \text{ semi-algebraic}\}.$$

   (notice: here $\mathrm{Spec}(A)$ is the type space $S_n(\mathbb{R})$).

7. The case of two points

Let A be a real closed ring which has exactly two prime ideals $\mathfrak{p}, \mathfrak{q}$. If there is no inclusion between $\mathfrak{p}$ and $\mathfrak{q}$, then $\mathrm{Spec}(A)$ is boolean. So assume $\mathfrak{p} \subseteq \mathfrak{q}$. Then A is a local domain with maximal ideal $\mathfrak{q}$ and the set $V = \{b \in \mathrm{qf}(A) \mid b \cdot \mathfrak{q} \subseteq \mathfrak{q}\}$ is the largest convex valuation ring of $\mathrm{qf}(A)$ lying over $(A, \mathfrak{q})$. In a diagram:

$$\begin{tikzcd} A \arrow[r] \arrow[d, two heads] & V \arrow[r] \arrow[d, two heads] & \mathrm{qf}(A) \\ A/\mathfrak{q} \arrow[r] & V/\mathfrak{m} & \end{tikzcd}$$

A complete list of axioms of A is given by saying:

(a) A is a real closed ring and a local domain

(b) the maximal ideal of A is convex in $\mathrm{qf}(A)$ (so actually $\mathfrak{m} = \mathfrak{q}$ above)

(c) the theory of the pair $A/\mathfrak{q} \subseteq V/\mathfrak{m}$ of real closed fields.

This follows from AKE-Delon. Further, it is easy to see that in fact every pair of real closed fields occurs as a pair from a real closed ring satisfying these axioms in the way shown (but in general $\dim(A) \geq 1$). Now pairs of real closed fields may or may not be decidable. A decidable pair is, e.g., the pair $\mathbb{R}_{\mathrm{alg}} \subseteq \mathbb{R}$. Undecidable pairs have been shown to exist by Baur and Macintyre.

8. The case of finitely many points

Let A be a real closed ring that has a finite spectrum (it is worth mentioning that each finite root system actually occurs among such spectra). We will stay descriptive here and indicate work in progress.

Since A has finite spectrum, we can build up A from real closed rings with 2 points in their spectrum (as on the previous slide), using finitely many standard constructions from commutative algebra, most notably fibre products. The way A is constructed in this sense can be read off the first order theory of A and it is hoped that a complete list of axioms for A is

1. naming this configuration
2. listing the theories of all pairs of real closed fields in the configuration.

9. Global functions - Dimension ≥ 2

Let R be a real closed field.

Theorem. (Gregorczyk)
If $X \subseteq R^n$ is of dimension ≥ 2, then the ring $C_{s.a.}(X, R)$ is undecidable: the lattice of closed s.a. subsets of X interprets arithmetic and this lattice is interpretable in the ring.

Theorem. (Tr.)
If $X \subseteq R^n$ is connected and of constant dimension ≥ 2, then the ring $C_{s.a.}(X, R)$ defines $\mathbb{Z}$. Consequently, if $R = \mathbb{R}$ and S is a real closed field, properly containing $\mathbb{R}$, then $C_{s.a.}(X, \mathbb{R})$ is not elementary equivalent to $C_{s.a.}(X_S, S)$.

Theorem. (Consequence of a theorem of V. Astier and the above)
The ring $C_{s.a.}(R^n, R)$ is not uniformly interpretable in the lattice of closed s.a. subsets of R^n (answer to a question of L. Darnière)

Similarly, rings of two-dimensional continuous s.a. germs define the integers.

10. **Global functions - Dimension $= 1$**

Open Problem: Is the ring $A = C_{s.a}(\mathbb{R}, \mathbb{R})$ decidable?

Remarks:

1. There are no local obstructions to decidability as in the case of dimension ≥ 2 (locally, A is a fibre product of two valuation rings, which has a decidable theory)
2. There is evidence that the vector lattice A (forget multiplication but keep scalar multiplication and the partial order) is decidable.
3. There is strong evidence that the ring of bounded s.a. (not necessarily continuous) functions $\mathbb{R} \longrightarrow \mathbb{R}$ is decidable.
4. On the negative side, A seems to be quite powerful: A interprets the ring D of all s.a. functions $\mathbb{R} \longrightarrow \mathbb{R}$; The family of all subrings of D that are finitely generated over A is a definable family in A.

The End

References (for real closed rings) I

Vincent Astier.
Elementary equivalence of lattices of open sets definable in o-minimal expansions of fields, 2011.
preprint.

Walter Baur.
Die Theorie der Paare reell abgeschlossener Körper.
In *Logic and algorithmic (Zurich, 1980)*, volume 30 of *Monograph. Enseign. Math.*, pages 25–34. Univ. Genève, Geneva, 1982.

Walter Baur.
On the elementary theory of pairs of real closed fields. II.
J. Symbolic Logic, 47(3):669–679, 1982.

Alexander Prestel and Niels Schwartz.
Model theory of real closed rings.
In *Valuation theory and its applications, Vol. I (Saskatoon, SK, 1999)*, volume 32 of *Fields Inst. Commun.*, pages 261–290. Amer. Math. Soc., Providence, RI, 2002.

Niels Schwartz.
Real closed rings.
In *Algebra and order (Luminy-Marseille, 1984)*, volume 14 of *Res. Exp. Math.*, pages 175–194. Heldermann, Berlin, 1986.

References (for real closed rings) II

Niels Schwartz.
The basic theory of real closed spaces.
Mem. Amer. Math. Soc., 77(397):viii+122, 1989.

Niels Schwartz and James J. Madden.
Semi-algebraic function rings and reflectors of partially ordered rings, volume 1712 of *Lecture Notes in Mathematics*.
Springer-Verlag, Berlin, 1999.

Marcus Tressl.
Super real closed rings.
Fund. Math., 194(2):121–177, 2007.

Marcus Tressl.
Bounded super real closed rings.
In *Logic Colloquium 2007*, volume 35 of *Lect. Notes Log.*, pages 220–237. Assoc. Symbol. Logic, La Jolla, CA, 2010.

Nicolas Guzy and Marcus Tressl.
pd-adically closed rings.
Extended abstract. Séminaire de Structures Algbriques Ordonnes 2008.
`http://personalpages.manchester.ac.uk/staff/Marcus.Tressl/papers/pAdicClosedRingDDG.pdf`