Group automorphisms: a dynamical point of view

Can you describe the space of compact group automorphisms
modulo dynamical equivalences?

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The setting

A continuous group automorphism $T : G \to G$ of a compact metric abelian group is a simple example of a dynamical system in several ways:

- Haar measure $m = m_G$ is defined on the Borel σ-algebra $\mathscr{B}$ and is preserved by T, so $(G, \mathscr{B}, m, T)$ is a measure-preserving dynamical system.
- G has a metric d, and T is continuous, so (G, d, T) is a topological dynamical system.

It is a special dynamical system in many ways, including:

- the algebraic structure is 'rigid';
- perturbation does not make sense;
- the dynamics is as homogeneous as possible (it locally looks the same everywhere).

More generally, if Γ is a discrete group with a homomorphism to the group of continuous automorphisms of G, then we can think of the action of Γ as a measurable (or topological) Γ-action T denote $T_\gamma : G \to G$ for each $\gamma \in \Gamma$.

Warnings: 1) These systems are very different to the systems found in 'homogeneous dynamics' (rotations on quotients of Lie groups by lattices) even in the case of a uniform lattice.

2) To avoid degeneracies, we always assume the action is 'ergodic' (there are no invariant L^2 functions $\equiv$ the dual automorphism is aperiodic $\equiv$ no iterate of T looks like the identity on part of the space).

The general problem

Let $\mathcal{G}$ denote the collection of all pairs (G, T), with G a compact metric abelian group and T a continuous automorphism ($\mathbb{Z}$-action) or a continuous Γ action.

Let $\sim$ be a dynamically meaningful notion of equivalence...

Describe the space $\mathcal{G}/\sim$.

Three natural equivalences

If (G_1, T_1) and (G_2, T_2) are two systems, then one notion of an equivalence between them is a commutative diagram

$$\begin{array}{ccc} G_1 & \xrightarrow{T_1} & G_1 \\ \phi\downarrow & & \downarrow\phi \\ G_2 & \xrightarrow[T_2]{} & G_2 \end{array}$$

where...

ϕ is a continuous isomorphism of groups (*algebraic isomorphism*),

ϕ is a homeomorphism (*topological conjugacy*), or

ϕ is an almost-everywhere defined isomorphism of the measure spaces (*measurable isomorphism*).

Clearly algebraic isomorphism $\implies$ topological conjugacy.

Less clearly, topological conjugacy $\implies$ measurable isomorphism.

Algebraic isomorphism

Part of this is not really dynamical at all: it is about classifying compact groups up to isomorphism, and understanding conjugacy classes in the automorphism group.

Example: Fix $G = \mathbb{T}^2$, the 2-torus. An automorphism corresponds to an element of $GL_2(\mathbb{Z})$. Now $A, B \in GL_2(\mathbb{Z})$ are conjugate only if they share determinant and trace, but that is not sufficient.

For instance, it is easy to check that

$$\begin{pmatrix} 3 & 10 \\ 1 & 3 \end{pmatrix}$$

is not conjugate to

$$\begin{pmatrix} 3 & 5 \\ 2 & 3 \end{pmatrix}.$$

A third invariant is needed, and this may be described in several ways:

- as an element of the ideal class group of the splitting field of the characteristic polynomial (Latimer & MacDuffee, 1933; extends to d-torus);
- as intersection information on images of closed curves $\equiv$ binary quadratic forms (Adler, Tresser, Worfolk, 1997; specific to 2-torus);
- as a rotation number (Adler, Tresser, Worfolk, 1997; specific to 2-torus).

This equivalence also has a *local version* (Baake, Roberts, Weiss, 2008) specific to the 2-torus.

Topological conjugacy

Example: Let G and H be finite groups, and consider the shift automorphisms on the compact groups $G^{\mathbb{Z}}$ and $H^{\mathbb{Z}}$.

Then topological conjugacy $\iff |G| = |H|$.

That is, all structure of the alphabet group is lost under topological conjugacy.

So on zero-dimensional groups, topological conjugacy has large equivalence classes.

On connected groups, the opposite happens: the topological structure is ‘rigid’.

Example: Assume that we have a topological conjugacy of toral automorphisms,

$$\begin{array}{ccc} \mathbb{T}^d & \xrightarrow{A} & \mathbb{T}^d \\ \phi \downarrow & & \downarrow \phi \\ \mathbb{T}^e & \xrightarrow[B]{} & \mathbb{T}^e \end{array}$$

We must have $d = e$ as dimension is a topological invariant, but much more is true.

Apply $\pi_1(\cdot)$ to linearize

$$\begin{array}{ccc} \mathbb{Z}^d & \xrightarrow{A} & \mathbb{Z}^d \\ {\scriptstyle \pi_1(\phi)}\downarrow & & \downarrow{\scriptstyle \pi_1(\phi)} \\ \mathbb{Z}^d & \xrightarrow[B]{} & \mathbb{Z}^d \end{array}$$

This means that A and B must be conjugate in the group $GL_d(\mathbb{Z})$ – algebraic isomorphism.

More is true: using Čech homology with coefficients in $\mathbb{T}$ gives the same result for automorphisms of *solenoids* (projective limits of tori).

In fact *much more* is true: the conjugacy ϕ itself must be a linear automorphism composed with rotation by a fixed point (Adler & Palais, 1965 for tori; Clark & Fokkink for solenoids).

The topological structure is surprisingly subtle. An obvious topological invariant to use is the dynamical zeta function,

$$\zeta_T(z) = \exp \sum_{n \geqslant 1} \frac{z^n}{n} |\{g \in G \mid T^n g = g\}|.$$

On zero-dimensional groups we again discern very little from this topological invariant. For the shift automorphism S on $G^{\mathbb{Z}}$, we have $\zeta_S(z) = \frac{1}{1-|G|z}$.

On connected groups we expect to do better, but life is not so simple.

Example: There are uncountably many topologically distinct 1-dimensional solenoidal automorphisms with zeta function $\frac{1-z}{1-2z}$ (Miles).

The point is that $\mathbb{Q}$ has many different subgroups.

Measurable isomorphism

This is an opaque equivalence because only the measurable structure is preserved – and any infinite compact abelian metric group is measurably isomorphic to $\mathbb{T}$.

Theorem: If T is ergodic, then T is measurably isomorphic to a Bernoulli shift (Katznelson, Lind, Miles & Thomas, Aoki).

That is, there is a countable partition ξ of G into measurable subsets so that:

1. ξ generates under T (the smallest T-invariant σ-algebra containing ξ is $\mathscr{B}$);and
2. ξ is independent under T: for any $n_1 < n_2 < \cdots < n_{k+1}$ we have
$$T^{n_1}\xi \vee \cdots \vee T^{n_k}\xi \perp T^{n_k+1}\xi.$$

Equivalently, T is measurably the same as a fair coin toss, or the shift map S on $A^{\mathbb{Z}}$ where A is the index set of ξ, with measure being the IID measure given on each coordinate by the probability vector $(m(B_a))_{a\in A}$, where $\xi = \{B_a \mid a \in A\}$.

A deep fact is that the Bernoulli shifts are classified in terms of their entropy:

$$h(S) = -\sum_{a \in A} m(B_a) \log m(B_a).$$

Theorem: Two Bernoulli shifts of the same entropy are measurably isomorphic (Ornstein).

Unfortunately this does not help as much as we might expect – it tells us that $\mathcal{G}/\sim$ for measurable isomorphism embeds into $\mathbb{R}_{>0}$, but it does not tell us more than that.

There is a Bernoulli shift for any given positive entropy – but is there a group automorphism?

Definition: The entropy of a group automorphism T is the rate of decay of volume of a Bowen-Dinaburg ball:

$$h(T) = \lim_{\epsilon \searrow 0} \lim_{n \to \infty} -\frac{1}{n} \log m \left(\bigcap_{i=0}^{n-1} T^{-i} B_\epsilon(0) \right).$$

This coincides with the entropy of the Bernoulli shift it is

Yuvinzkii's formula

Imagine a toral automorphism has eigenvalues λ_i with

$$|\lambda_1| \leqslant \cdots \leqslant |\lambda_s| \leqslant 1 < |\lambda_{s+1}| \leqslant \cdots \leqslant |\lambda_d|.$$

Then we think $\bigcap_{i=0}^{n-1} B_\epsilon(0)$ will have Haar volume

$$C\epsilon^d \left(\prod_{i=s+1}^{d} |\lambda_i| \right)^{-(n-1)} .$$

So we expect

$$h(T) = \sum_{i=s+1}^{d} \log |\lambda_i| = \sum_{i=1}^{d} \log^+ |\lambda_i|$$

... which can be written

$$h(T) = \int_0^1 \log|f(\mathrm{e}^{2\pi\mathrm{i}t})|\mathrm{d}t$$

(by Jensen's theorem), the *Mahler measure* $m(f)$ of f, the characteristic polynomial.

This is really a localization or linearization, and adeles can be used to make a similar calculation for solenoids.

Theorem: In general, $h(T) = \log k + A$, where $k \in \mathbb{N} \cup \{\infty\}$ and A lies in the closure of the set $\{m(f) \mid m(f) > 0\}$ (Yuzvinskii).

Lehmer's problem: Is

$$\inf\{m(f) \mid m(f) > 0\} > 0?$$

If the answer is yes, then $\mathcal{G}/\sim$ is *countable*.

If the answer is no, then entropy defines a bijection $\mathcal{G}/\sim \longrightarrow \mathbb{R}_{>0}$.

Seeking continua...

Perturbations don't exist, Lehmer's problem is difficult, so are there continua at all? Or is $\mathcal{G}$ inherently granular (discrete)?

Theorem: For any $C \in [0, \infty]$ there is a compact group automorphism $T : X \to X$ with

$$\frac{1}{n} \log |\{x \in X \mid T^n x = x\}| \to C$$

as $n \to \infty$.

So the invariant 'logarithmic growth rate of periodic points if it exists' has $[0, \infty]$ as a fibre.

But: the examples are zero-dimensional (not too bad), non-ergodic (this is really cheating), and quite baffling (the construction uses Linnik's theorem on appearance of primes in arithmetic progressions).

We don't understand if the exponential growth rate of periodic points on connected groups exhibits a continuum, and this is probably a disguised form of Lehmer's problem.

With more smoothing we can do better. Let

$$M_T(N) = \sum_{|\tau| \leqslant N} \frac{1}{e^{h|\tau|}},$$

where $|\tau|$ denotes length of a closed orbit τ, and h is topological entropy.

Paradigm: For $T : x \mapsto 2x \bmod 1$ (not quite an automorphism, but a handy example), we have:

- $2^n - 1$ points fixed by T^n and topological entropy $\log 2$;
- hence $2^n/n + O(2^{n/2})$ closed orbits of length n;
- hence $M_T(N)$ is more or less $\sum_{n \leqslant N} \frac{2^n/n}{2^n} \sim \log N$.

It turns out that many group automorphisms have

$$M_T(N) \sim \kappa \log N$$

(and in some cases more refined asymptotics are also known).
Baier, Jaidee, Stephens, Ward find some continua.

Theorem: For any $\kappa \in (0, 1)$ there is an ergodic compact connected group automorphism $T : X \to X$ with $M_T(N) \sim \kappa \log N$.

Theorem: For any $r \in \mathbb{N}$ and $\kappa > 0$ there is an ergodic compact connected group automorphism $T : X \to X$ with $M_T(N) \sim \kappa(\log \log N)^r$.

Theorem: For any $\delta \in (0, 1)$ and $k > 0$ there is an ergodic compact connected group automorphism $T : X \to X$ with $M_T(N) \sim k(\log N)^\delta$.

Constructions in 1-solenoids

The simplest connected groups are the one-dimensional solenoids, which are in 1-to-1 correspondence with subgroups of $\mathbb{Q}$. These are easy to describe (*unlike the subgroups of* $\mathbb{Q}^2$).

The simplest of these are the subrings: take S a set of primes, and (say) the map $x \mapsto 2x$ on

$$\{r = \tfrac{a}{b} \mid p|b \implies p \in S\}.$$

Dualizing gives a group endomorphism with

$$|\{x \in X \mid T^n x = x\}| = (2^n - 1) \prod_{p \in S} |2^n - 1|_p.$$

So the construction boils down to statements about sets of primes.

The wider picture

Replacing a single automorphism with a Γ action T produces even more rigid systems because the conjugacies are having to intertwine more maps.

Entirely new phenomena emerge, for example *abelian measurable rigidity*.

Theorem: For $d \geqslant 2$, any measurable isomorphism between expansive, mixing, irreducible (closed invariant sets are finite) $\mathbb{Z}^d$-actions by automorphisms is an affine map (Kitchens & Schmidt; Katok & Spatzier).

Example: There exist mixing $\mathbb{Z}^8$-actions by automorphisms that do not exhibit this rigidity (Bhattacharya). (They are not irreducible)

Some old phenomena survive, for example *topological rigidity*.

Theorem: For $\mathbb{Z}^d$-actions by automorphisms ($d \geqslant 1$) of compact connected groups which are mixing and satisfy a descending chain condition on closed invariant subgroups, any equivariant continuous map must be affine (topological rigidity) if and only if the entropy of the target system is finite.