The classification problem for Toeplitz subshifts

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Standard Borel Spaces

Definition

- A topological space (X, τ) is called a Polish space if it is separable and completely metrizable.
- A measurable space (X, Ω) is called a standard Borel space if Ω is the Borel σ-algebra $\mathcal{B}(\tau)$ of some Polish topology τ on X.
- E.g. $\mathbb{R}$, $[0,1]$, 2^ω, ω^ω,$[0,1]^\omega$

Definition

Let X, Y be standard Borel spaces.

- A map $\varphi : X \to Y$ is called Borel if it is measurable, i.e. $f^{-1}[B]$ is Borel for all Borel subsets $B \subseteq Y$.
- Equivalently, $\varphi : X \to Y$ is Borel if *graph*(φ) is a Borel subset of $X \times Y$

Polish Spaces vs. Standard Borel Spaces

Theorem

A subspace S of a Polish space X is Polish if and only if it is a G_δ subset of X.

This means that we cannot pass to some arbitrary subspace if we want to keep the induced topology same and Polish. On the other hand:

Theorem

Let (X, τ) be a Polish space and $S \subseteq X$ be any Borel subset. Then there exists a Polish topology $\tau_S \supseteq \tau$ on X such that $\mathcal{B}(\tau_S) = \mathcal{B}(\tau)$ and S is clopen in τ_S.

Corollary

If $(X, \mathcal{B})$ is a standard Borel space and $Y \in \mathcal{B}$, then $(Y, \mathcal{B} \restriction Y)$ is a standard Borel space.

The Borel Isomorphism Theorem

Definition

Two standard Borel spaces (X, Ω_1) and (Y, Ω_2) are called isomorphic if there exists a bimeasurable bijection between X and Y.

A bimeasurable version of Schroder-Bernstein theorem holds for standard Borel spaces. For any uncountable standard Borel space X, by embedding 2^ω into X, X into $[0,1]^\omega$ and $[0,1]^\omega$ into 2^ω in a bimeasurable way, we have:

Theorem (Kuratowski)

Any two uncountable standard Borel spaces are isomorphic.

Coding countably infinite first-order structures into Polish spaces

Let $\mathcal{L} = \{R_i : i \in I\}$ be a countable language where R_i is an n_i-ary relation symbol and let $X_{\mathcal{L}} = \prod_{i \in I} 2^{\omega^{n_i}}$. Then $X_{\mathcal{L}}$ is a Polish space elements of which code $\mathcal{L}$-structures with universe ω as follows. For any $x = (x_i)_{i \in I} \in X_{\mathcal{L}}$, the structure

$$M_x = (\omega, \{R_i^x\}_{i \in I})$$

represented by x is defined by:

$$R_i^x(k_1, ..., k_{n_i}) \Leftrightarrow x_i(k_1, ..., k_{n_i}) = 1$$

Example

If we let $\mathcal{L}$ consist of a single binary relation E, then the Polish space $2^{\omega \times \omega}$ codes the space of countable graphs with underlying set ω. For any such "graph" $x \in 2^{\omega \times \omega}$, there is an edge between the vertices i and j if and only if $x(i,j) = 1$

Coding countably infinite first-order structures into Polish spaces

Remark

If we consider the infinite symmetric group $Sym(\omega)$ *as a subspace of the Baire space* ω^ω*, it becomes a Polish group with a natural Borel action on* $X_{\mathcal{L}}$*. Then* $x, y \in X_{\mathcal{L}}$ *are in the same* $Sym(\omega)$*-orbit if and only if* $M_x \cong M_y$*.*

Given any $\mathcal{L}_{\omega_1,\omega}$ sentence ψ, the class of all structures with underlying set ω that models ψ, $Mod(\psi) = \{x \in X_{\mathcal{L}} : M_x \models \psi\}$ is an isomorphism-invariant Borel subset of $X_{\mathcal{L}}$.

Example

Let $\mathcal{L}$ consist of a single ternary relation. If we associate any countable group $(\omega, \cdot)$ with the characteristic function of $\cdot \subseteq \omega \times \omega \times \omega$, then the class of countable groups, being axiomatized by a $\mathcal{L}_{\omega_1,\omega}$-sentence, is a Borel subset of $X_{\mathcal{L}}$ and thus itself is a standard Borel space.

The Isomorphism Relation on $Mod(\psi)$

The isomorphism relation on $Mod(\psi)$ is given by

$$x \cong y \Leftrightarrow \exists g \in Sym(\omega)\ g \cdot x = y$$

and is an analytic equivalence relation being the projection of graph of a Borel action, and need not be Borel in general.

Example (Mekler)

The isomorphism relation on the space of countable groups $\cong_{\mathcal{G}}$ is not Borel.

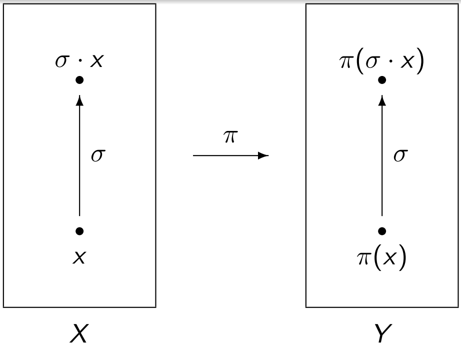
On the other hand, for the structures that are of "finite rank" in a broad sense, the isomorphism relation is a Borel relation. E.g. Finitely generated groups, finite rank torsion-free abelian groups, connected locally finite graphs,...

An example from topological dynamics

Fix some $n \geq 2$.

Definition

- A closed infinite subset S of the Cantor space $n^{\mathbb{Z}}$ is called a subshift if it is invariant under the shift operator $(\sigma(x))(k) = x(k+1)$.
- Two subshifts S and T are called topologically conjugate if there exists a homeomorphism $\psi : S \to T$ such that $\psi \circ \sigma = \sigma \circ \psi$



Standard Borel Space of Subshifts

Definition

Let X be a Polish space and $K(X)$ be the set of all non-empty compact subsets of X. Then the Vietoris topology on $K(X)$ generated by the sets $\{K \in K(X) : K \subseteq U\}$ and $\{K \in K(X) : K \cap U \neq \emptyset\}$ for U open in X is a Polish topology. If d is a complete metric on X inducing its Polish topology, then the Hausdorff metric

$$\delta_H(K, L) = max\{max_{x \in K} d(x, L), max_{x \in L} d(x, K)\}$$

is a compatible metric for the Vietoris topology.

Theorem

The collection $\mathcal{S}_n$ of subshifts of $n^{\mathbb{Z}}$ is a Borel subset of $K(n^{\mathbb{Z}})$, and hence is a standard Borel space and the topological conjugacy relation on it is a Borel equivalence relation.

Borel Equivalence Relations

Definition

- Let X be a standard Borel space. An equivalence relation $E \subseteq X^2$ is called Borel if it is a Borel subset of $X \times X$. A Borel equivalence relation is called countable if every E-equivalence class is countable.
- Let G be a Polish group. A standard Borel G-space is a standard Borel space X equipped with a Borel G-action. The corresponding orbit equivalence relation is denoted by E_G^X.

Example

Let G be a countable group endowed with discrete topology and X be a standard Borel G-space. Then, E_G^X is a countable Borel equivalence relation.

Borel Reducibility

Definition

Let E, F be Borel equivalence relations on standard Borel spaces X and Y respectively.

- We say E is Borel reducible to F, denoted by $E \leq_B F$, if there exists a Borel map $f : X \to Y$ such that for all $x, y \in X$

  $$x \; E \; y \Leftrightarrow f(x) \; F \; f(y)$$

  In this case, f is said to be a reduction from E to F.
- $E \sim_B F$ if both $E \leq_B F$ and $F \leq_B E$.
- $E <_B F$ if $E \leq_B F$ but $F \not\leq_B E$.

If E is Borel reducible to F, then the classification with respect to E is, intuitively speaking, no harder than the classification with respect to F. The intuition behind the requirement that f is Borel is that Borel maps are thought as "explicit computations".

Climbing up in the hierarchy of $\leq_B$

Theorem (Silver)

Let E be a Borel equivalence relation on a standard Borel space. Then either $E \leq_B \Delta_\omega$ or $\Delta_{2^\omega} \leq_B E$.

Definition

A Borel equivalence relation E is called smooth if $E \leq_B \Delta_X$ for some (equivalently every) uncountable standard Borel space X.

Example

Let $\cong_\psi$ denote the isomorphism relation on the standard Borel space of countable divisible abelian groups. Any countable divisible abelian group G can be written as $(\oplus_{i \in r_0(G)} \mathbb{Q}) \oplus (\oplus_{p \in \mathbb{P}} \oplus_{j \in r_p(G)} \mathbb{Z}[p^\infty])$ where $0 \leq r_0(G), r_p(G) \leq \omega$ and these ranks determine G up to isomorphism. Then, the Borel map $f(G) = (r_0(G), r_2(G), r_3(G), \ldots)$ witnesses the fact that $\cong_\psi$ is smooth.

Examples of non-smooth Borel equivalence relation

Example

Let E_0 be the countable Borel equivalence relation on 2^ω defined by:

$$x \; E_0 \; y \Leftrightarrow \exists n \, \forall m \geq n \; x(m) = y(m)$$

Assume that there is a Borel reduction $f : 2^\omega \to [0,1]$ from E_0 to $\Delta_{[0,1]}$. If we endow 2^ω with its usual product probability measure, then both $f^{-1}[0,1/2]$ and $f^{-1}[1/2,1]$ are Borel tail events, and one of them has to have measure 1 by Kolmogorov 0-1 law. Continuing in this manner, we see that f is constant almost everywhere, which is a contradiction.

It turns out that E_0 is the immediate successor of Δ_{2^ω} with respect to $\leq_B$

Theorem (Harrington-Kechris-Louveau)

Let E be a Borel equivalence relation on a standard Borel space. Then either $E \leq_B \Delta_{2^\omega}$ or $E_0 \leq_B E$.

The Feldman-Moore Theorem

Recall that any countable discrete group G acting a standard Borel G-space induces a countable Borel equivalence relation as its orbit equivalence relation. Remarkably, the converse of this is also true:

Theorem (Feldman-Moore)

Let E be a countable Borel equivalence relation on a standard Borel space X. Then, there exists a countable discrete group G and a Borel G-action on X such that $E = E_G^X$. Moreover, G can be chosen such that

$$x \; E \; y \Leftrightarrow \exists g \in G \; g^2 = 1 \; \wedge \; g \cdot x = y$$

An important consequence of this theorem is that, up to $\sim_B$, there is a $\leq_B$-maximal element for countable Borel equivalence relations.

Theorem (Dougherty-Jackson-Kechris)

There exists a universal countable Borel equivalence relation E_ω, i.e. for all countable Borel equivalence relations E we have $E \leq_B E_\omega$.

Examples of universal countable Borel equivalence relations

Let $\mathbb{F}_2$ be the free group on two generators and E_∞ be the orbit equivalence relation of the Borel action of $\mathbb{F}_2$ on $2^{\mathbb{F}_2}$ defined by $(g \cdot x)(h) = x(g^{-1}h)$.

Theorem (Dougherty-Jackson-Kechris)

E_∞ is a universal countable Borel equivalence relation.

Theorem (Clemens, 2009)

Topological conjugacy on the space of subshifts $\mathcal{S}_n$ is a universal countable Borel equivalence relation.

Theorem (Thomas-Velickovic, 1998)

The isomorphism relation on the space of finitely generated groups $\mathcal{G}_{fg}$ is a universal countable Borel equivalence relation.

Cantor Minimal Systems

Definition

Let (X, T) be a Cantor topological dynamical system, i.e. X is a Cantor set and $T : X \to X$ is a homeomorphism. (X, T) is called a Cantor minimal system if X has no non-empty closed T-invariant proper subsets.

Definition

A point $x \in X$ is called *almost periodic* (or *uniformly recurrent*) if for every open neighborhood U of x, the set $R = \{i \in \mathbb{Z} : T^i(x) \in U\}$ of returning times has bounded gaps, i.e. there exists $l \geq 1$ such that for all $n \in \mathbb{Z}$, $R \cap \{n, n+1, ..., n+l\} \neq \emptyset$.

Remark

X is minimal if and only if X is the orbit closure $\bar{O}(x) = \overline{\{T^n(x) : n \in \mathbb{Z}\}}$ *of some almost periodic point x, in which case every point is almost periodic.*

Topological Full Groups of Cantor Minimal Systems

Definition

Let (X, T) be a Cantor minimal system. The topological full group $[[T_X]]$ is the group of homeomorphisms $\pi : X \to X$ such that there exists a clopen partition $X = C_1 \sqcup C_2 \sqcup ... \sqcup C_m$ and $n_1, n_2, ..., n_m \in \mathbb{Z}$ such that $\pi \restriction C_i = T^{n_i} \restriction C_i$ for all $1 \leq i \leq m$.

Definition

Let (X, T) and (Y, S) be Cantor minimal systems. (X, T) and (Y, S) are topologically conjugate if there exists a homeomorphism $\pi : X \to Y$ such that $\pi \circ T = S \circ \pi$. (X, T) and (Y, S) are said to be flip conjugate if (X, T) is topologically conjugate to either (Y, S) or (Y, S^{-1}).

Topological Full Groups of Cantor Minimal Systems

Theorem (Giordano-Putnam-Skau and Bezuglyi-Medynets)

Let (X, T) and (Y, S) be Cantor minimal systems. The following are equivalent:
- *(X, T) and (Y, S) are flip conjugate.*
- *$[[T_X]]$ and $[[S_Y]]$ are isomorphic.*
- *$[[T_X]]'$ and $[[S_Y]]'$ are isomorphic.*

Combining the work of Matui and Juschenko-Monod on topological full groups, we have the following theorem:

Theorem

If (X, T) is topologically conjugate to a minimal subshift over a finite alphabet, then $[[T_X]]'$ is an infinite finitely generated simple amenable group.

Back to the space of finitely generated groups

Theorem (Thomas, 2012)

The isomorphism relations on the space of finitely generated simple groups and finitely generated amenable groups are not smooth.

The idea is to construct a Borel reduction from flip conjugacy on minimal subshifts to isomorphism on finitely generated simple amenable groups and then to show that flip conjugacy on minimal subshifts is not smooth. It follows from the Feldman-Moore theorem that if $E \subseteq F$ are countable Borel relations and E is not smooth, then F is not smooth. Hence, it is sufficient to prove that topological conjugacy on minimal subshifts is not smooth.

Toeplitz Sequences

Definition

An element $\alpha \in n^{\mathbb{Z}}$ is called a *Toeplitz sequence* if for all $i \in \mathbb{Z}$ there exists $j \in \mathbb{N}^+$ such that $\alpha(i + kj) = \alpha(i)$ for all $k \in \mathbb{Z}$. Equivalently, Toeplitz sequences are those in which every block appears periodically.

Example

Let $*$ denote the blank symbol. Construct a sequence of bisequences over $n \cup \{*\}$ inductively as follows:
- Start with the bisequence with constant value *.
- At the k-th stage of the construction, periodically fill one hole that is not yet filled in each interval $[m2^k, (m+1)2^k)$.
- Alternate the symbol to be used every stage.
- Alternate the relative position of the hole to be filled every stage, that is, if the hole to be filled is chosen from the left half of the interval at stage k, then the hole to be filled at the $(k+1)$-th stage will be chosen from the right half of the corresponding interval.

Example (Continued)

The bisequence obtained by taking the limit of the following bisequences will be a Toeplitz sequence.

...**...

...0*0*0*0*0*0*0*0*0*0*0*0*0*0*0*0*0*0*0*0*0*0*0*0...

...0*010*010*010*010*010*010*010*010*010*010*010*010*010...

...00010*0100010*0100010*0100010*0100010*0100010*0100010*010...

...0001010100010*01000101010001 0*01000101010001 0*01000101010...

...0001010100010*010001010100010001000101010001 0*01000101010... and so on.

Conversely, every Toeplitz sequence can be imagined to be obtained by such a recursive construction where one chooses a period length k at each stage and periodically fill some of the holes not yet filled in each interval $[mk, (m+1)k)$ in such a way that every blank symbol is filled at some stage of the construction.

Toeplitz Sequences

To each Toeplitz sequence $\alpha \in n^{\mathbb{Z}}$, we will associate the following objects. For each $p \in \mathbb{N}^+$,

Definition

- For each symbol $a \in n$, set $Per_p(\alpha, a) = \{i \in \mathbb{Z} : \forall l \in \mathbb{Z}\ \alpha(i + pl) = a\}$
- The *p-periodic parts* of α is defined to be the set of indices

$$Per_p(\alpha) = \bigcup_{a \in n} Per_p(\alpha, a) = \{i \in \mathbb{Z} : \forall l \in \mathbb{Z}\ \alpha(i) = \alpha(i + pl)\}$$

- The sequence obtained from α by replacing $\alpha(i)$ with a new blank symbol * for each $i \notin Per_p(\alpha)$ will be called the *p-skeleton* of α.
- The *p-symbols* of α is the set of blocks $W_p(\alpha) = \{\alpha[ip, (i+1)p) : i \in \mathbb{Z}\}$

Toeplitz Sequences

Definition

A positive integer $p \in \mathbb{N}^+$ is an *essential period* of α if $Per_p(\alpha) \neq \emptyset$ and for all $q<p$ we have $Per_p(\alpha) \neq Per_q(\alpha)$. Equivalently, p is an essential period of α if and only if p-skeleton of α is not periodic with any smaller period.

Definition

A *period structure* for a non-periodic Toeplitz sequence $\alpha \in n^{\mathbb{Z}}$ is an increasing sequence $(p_i)_{i\in\mathbb{N}}$ of natural numbers such that:

- For all $i \in \mathbb{N}$, p_i is an essential period of α and $p_i | p_{i+1}$,
- $\bigcup_{i\in\mathbb{N}} Per_{p_i}(\alpha) = \mathbb{Z}$

Toeplitz Subshifts

Any Toeplitz sequence $\alpha \in n^{\mathbb{Z}}$ is almost periodic in the closure of its orbit under the shift transformation since all of its subblocks are periodic. Thus its shift orbit closure $\bar{O}(\alpha)$ is a minimal subshift. We shall call these minimal subshifts *Toeplitz subshifts* (or *Toeplitz flows*).

Remark

The class of Toeplitz subshifts $\mathcal{T}_n$ over the alphabet n is a Borel subset of the standard Borel space of minimal subshifts $\mathcal{M}_n$ and hence is itself a standard Borel space. Similarly, we can form the standard Borel space $\mathcal{T}_n^p$ of pointed Toeplitz subshifts $\{(O, \alpha) : O$ is a Toeplitz subshift & $\alpha \in O$ & α is a Toeplitz sequence$\}$.

We can define a stronger notion of isomorphism on the space of pointed Toeplitz flows as follows: Two pointed Toeplitz flows (O, α) and (O', β) are said to be pointed topologically conjugate if there exists a topological conjugacy $\pi : O \to O'$ such that $\pi(\alpha) = \beta$.

Scale of a Toeplitz flow

For any sequence $(u_i)_{i\in\mathbb{N}}$ of natural numbers, define $\prod((u_i)_{i\in\mathbb{N}})$ to be the formal product $\mathbf{u} = \prod_{i\in\mathbb{N}^+} \mathrm{p}_i^{k_i}$ where p_i is the i-th prime number and $k_i \in \mathbb{N} \cup \{\infty\}$ such that

$$k_i = sup\{j \in \mathbb{N} : \exists m \in \mathbb{N}\ \mathrm{p}_i^j | u_m\}$$

If $\alpha \in n^{\mathbb{Z}}$ is a non-periodic Toeplitz sequence and $(u_i)_{i\in\mathbb{N}}$ is a period structure of α, we will refer to the supernatural number $\prod((u_i)_{i\in\mathbb{N}})$ as the *scale* of α and $(u_i)_{i\in\mathbb{N}}$ will be called a factorization of the scale $\prod((u_i)_{i\in\mathbb{N}})$.

Remark

The scale of a Toeplitz sequence α does not depend on the choice of its period structure and any two Toeplitz sequences in the same Toeplitz flow give the same scale. Topologically conjugate Toeplitz flows have the same scale. However, not all the flows having the same scale are isomorphic.

Complexity of Topological Conjugacy and Pointed Topological Conjugacy on Toeplitz Flows

Let E_{tc} and E_{ptc} denote the topological conjugacy and pointed topological conjugacy relations on the standard Borel spaces $\mathcal{T}_n$ and $\mathcal{T}_n^p$ respectively. It follows from the Curtis-Lyndon-Hedlund Theorem that E_{tc} and E_{ptc} are countable Borel equivalence relations.

Theorem (Thomas, 2012)

E_{tc} and E_{ptc} are not smooth, i.e. $E_0 \leq_B E_{tc}$ and $E_0 \leq_B E_{ptc}$.

Corollary

Flip conjugacy on the space of minimal subshifts is not smooth and hence the isomorphism relation on the space of infinite finitely generated simple amenable groups is not smooth.

Complexity of Topological Conjugacy and Pointed Topological Conjugacy on Toeplitz Flows

Theorem (Downarowicz-Kwiatkowski-Lacroix, 1995)

Let (O_1, α) and (O_2, β) be pointed Toeplitz flows with common scale where α and β are Toeplitz sequences and let $(r_t)_{t \in \mathbb{N}}$ be a factorization of their common scale. Then, (O_1, α) and (O_2, β) are pointed topologically conjugate if and only if there exist $t \in \mathbb{N}$ and a bijective function $\Gamma : W_{r_t}(\alpha) \to W_{r_t}(\beta)$ such that $\beta[kr_t, (k+1)r_t) = \Gamma(\alpha[kr_t, (k+1)r_t))$ for all $k \in \mathbb{Z}$.

Using this criterion, for each scale, one can construct a countable locally finite group action inducing the pointed topological conjugacy relation for the flows having that scale and it follows that

Theorem (K.)

$E_{ptc} \sim_B E_0$.

From Topological Conjugacy to Pointed Topological Conjugacy

Does there exist a Borel way to choose a Toeplitz sequence from each Toeplitz flow?

Theorem

There exists a Borel map $f : \mathcal{T}_n \to n^{\mathbb{Z}}$ such that $f(O) \in O$ is a Toeplitz sequence.

Is it possible to do this in an isomorphism invariant way? If there exists a Borel map $f : \mathcal{T}_n \to n^{\mathbb{Z}}$ such that $O \cong O'$ implies $(O, f(O)) \cong (O', f(O'))$, then obviously $E_{tc} \leq_B E_{ptc} \leq_B E_0$.

Conjecture (Common Sense)

There does not exist a Borel map $f : \mathcal{T}_n \to n^{\mathbb{Z}}$ such that $O \cong O'$ implies $(O, f(O)) \cong (O', f(O'))$.

Toeplitz Subshifts Admitting Single Hole Constructions

Definition

Let $\alpha \in n^{\mathbb{Z}}$ be an element from a Toeplitz flow O and $p \in \mathbb{N}^+$. If p is a period of α, that is, $Per_p(\alpha) \neq \emptyset$, then let $a, b \in \mathbb{N}^+$ be the least positive integers that satisfy

- $a \leq b$, $[a, b] \subseteq Per_p(\alpha)$ and,
- For all $a', b' \in \mathbb{Z}$, $|b - a| \geq |b' - a'|$ whenever $[a', b'] \subseteq Per_p(\alpha)$.

and define $\eta_p(\alpha) = |b - a| + 1$.

Definition

A Toeplitz flow O is said to admit a single hole construction if there exists a period structure $(u_i)_{i \in \mathbb{N}}$ such that $\eta_{u_i}(O) = u_i - 1$ for all $i \in \mathbb{N}$.

A successor to E_0

Let E_1 be the Borel equivalence relation on the standard Borel space $(2^\omega)^\omega$ defined by

$$xE_1y \Leftrightarrow \exists m \in \omega \ \forall n \ \geq m \ x_n = y_n$$

Theorem (Kechris-Louveau)

For every Borel equivalence relation E such that $E \leq_B E_1$, either $E \leq_B E_0$ or $E \sim_B E_1$. Moreover, E_1 is not Borel reducible to any countable Borel equivalence relation.

Hence, in order to show that a non-smooth countable Borel equivalence relation is Borel bireducible with E_0, it is sufficient to show that it is Borel reducible to E_1.

Topological Conjugacy on Toeplitz Subshifts Admitting Single Hole Constructions

Let $\mathcal{T}_n^*$ be the collection of Toeplitz flows that are isomorphic to Toeplitz flows admitting single hole constructions. It follows that $\mathcal{T}_n^*$ is a Borel subset of $\mathcal{T}_n$. Let E_{tc}^* be the countable Borel equivalence relation $E_{tc} \cap (\mathcal{T}_n^* \times \mathcal{T}_n^*)$.

Theorem (K.)

$E_{tc}^* \leq_B E_1$.

Corollary

$E_{tc}^* \sim_B E_0$.

Use of the Kechris-Louveau dichotomy in the proof of that $E_{tc}^* \leq_B E_0$ seems unavoidable. Find a Borel reduction from E_{tc}^* to E_0 on 2^ω explicitly or show that E_{tc}^* is hyperfinite by exhibiting an increasing sequence of finite Borel equivalence relations whose union is E_{tc}^*.