Quantum Groups, R-Matrices and Factorization

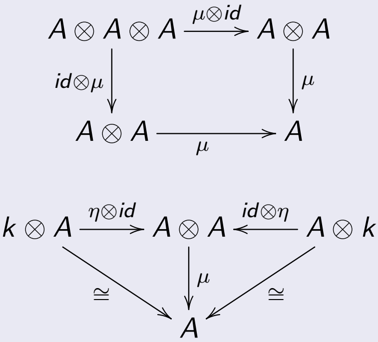
Münevver Çelik

5 November 2015

Algebra

Definition

Let A be a vector space over $\mathbb{k}$ and $\mu : A \otimes A \to A$ and $\eta : \mathbb{k} \to A$ be linear maps. The triple (A, μ, η) is said to be an algebra if the following diagrams commute:



Coalgebra

Definition

Let A be a vector space over $\mathbb{k}$ and $\Delta : A \to A \otimes A$ and $\varepsilon : A \to \mathbb{k}$ be linear maps. The triple (A, Δ, ε) is said to be a coalgebra if the following diagrams commute:

$$\begin{array}{ccc} A & \xrightarrow{\Delta} & A \otimes A \\ {\scriptstyle \Delta}\downarrow & & \downarrow{\scriptstyle id \otimes \Delta} \\ A \otimes A & \xrightarrow[\Delta \otimes id]{} & A \otimes A \otimes A \end{array}$$

$$\begin{array}{ccccc} k \otimes A & \xleftarrow{\varepsilon \otimes id} & A \otimes A & \xrightarrow{id \otimes \varepsilon} & A \otimes k \\ & {\scriptstyle \cong}\nwarrow & \uparrow{\scriptstyle \Delta} & \nearrow{\scriptstyle \cong} & \\ & & A & & \end{array}$$

Sweedler's sigma notation

Notation

(Sweedler's sigma notation) In order avoid the complexity of index notation we write

$$\Delta(x) = \sum_{(x)} x' \otimes x''$$

for any $x \in A$.

- If (A, μ, η) is an algebra then so is $(A \otimes A, \mu \otimes \mu, \eta \otimes \eta)$.

- If (A, μ, η) is an algebra then so is $(A \otimes A, \mu \otimes \mu, \eta \otimes \eta)$.
- If (A, Δ, ε) is a coalgebra then so is $(A \otimes A, (id \otimes \tau \otimes id) \circ (\Delta \otimes \Delta), \varepsilon \otimes \varepsilon)$, where $\tau(a \otimes b) = b \otimes a$.

Bialgebra

Definition

Let (A, μ, η) be an algebra and (A, Δ, ε) is a coalgebra. The quintuple $(A, \mu, \eta, \Delta, \varepsilon)$ is said to be a bialgebra if the maps μ and η are morphisms of coalgebras or equivalently, the maps Δ and ε are morphisms of algebras.

Hopf Algebra

Definition

Let $(H, \mu, \eta, \Delta, \varepsilon)$ be a bialgebra. An endomorphism S of H is called an antipode for the bialgebra H if

$$\sum_{(x)} S(x')x'' = \sum_{(x)} x'S(x'') = \varepsilon(x)1$$

for all $x \in H$.

A Hopf algebra is a bialgebra with an antipode.

R-matrix

Definition

Let V be a vector space. An automorphism c of $V \otimes V$ is called an R-matrix if it satisfies the Yang-Baxter equation

$$(c \otimes id_V)(id_V \otimes c)(c \otimes id_V) = (id_V \otimes c)(c \otimes id_V)(id_V \otimes c)$$

which holds in the automorphism group of $V \otimes V \otimes V$

Universal R-matrix

Definition

A bialgebra $(H, \mu, \eta, \Delta, \varepsilon)$ is called quasi-cocommutative if there exists an invertible element R of the algebra $H \otimes H$ such that for all $x \in H$ we have

$$\Delta^{op}(x) = R\Delta(x)R^{-1}.$$

Here $\Delta^{op} = \tau_{H,H} \circ \Delta$ where $\tau_{H,H}(h_1 \otimes h_2) = h_2 \otimes h_1$. R is called the universal R-matrix of the bialgebra H. A Hopf algebra is quasi-cocommutative if its underlying bialgebra is quasi-cocommutative.

Braided Hopf Algebra

Definition

A quasi-cocommutative bialgebra $(H, \mu, \eta, \Delta, \varepsilon, R)$ *or a quasi-cocommutative Hopf algebra* $(H, \mu, \eta, \Delta, \varepsilon, S, R)$ *is braided if the universal R-matrix satisfies the following relations:*

$$(\Delta \otimes id_H)(R) = R_{13}R_{23}$$

$$(id_H \otimes \Delta)(R) = R_{13}R_{12}.$$

R-matrix from a Braided Hopf Algebra

Let $(H, \mu, \eta, \Delta, \varepsilon, R)$ be a braided bialgebra and V be an H-module. The automorphism $c^R_{V,V}$ of $V \otimes V$ defined by

$$c^R_{V,V}(v \otimes w) = \tau_{V,V}[R(v \otimes w)]$$

is an R-matrix.

Module-coalgebra

Definition

Let $(H, \mu, \eta, \Delta_H, \varepsilon_H)$ *be a bialgebra and* $(C, \Delta_C, \varepsilon_C)$ *be a coalgebra.* C *is said to be a module-coalgebra over* H *if there exists a morphism of coalgebras* $\phi : H \otimes C \to C$ *inducing an* H*-module structure on* C*, that is,*

$$
\begin{aligned}
(\phi \otimes \phi)\Delta_{H\otimes C} &= \Delta_C \phi \\
\varepsilon_{H\otimes C} &= \varepsilon_C \phi \\
\phi(\mu \otimes id_C) &= \phi(id_H \otimes \phi) \\
\phi(\eta \otimes id_C) &= id_C
\end{aligned}
$$

Matched pair

Definition

A pair (X, A) of bialgebras is matched if there exist linear maps $\alpha : A \otimes X \to X$ and $\beta : A \otimes X \to A$ turning X into a module-coalgebra over A, and turning A into a right module-coalgebra over X, such that, if we set

$$\alpha(a \otimes x) = a \cdot x \quad \text{and} \quad \beta(a \otimes x) = a^x,$$

the following conditions are satisfied:

Definition

$$a \cdot (xy) = \sum_{(a)(x)} (a' \cdot x')(a''^{x''} \cdot y),$$

$$a \cdot 1 = \varepsilon(a)1,$$

$$(ab)^x = \sum_{(b)(x)} a^{b' \cdot x'} b''^{x''},$$

$$1^x = \varepsilon(x)1,$$

$$\sum_{(a)(x)} a'^{x'} \otimes a'' \cdot x'' = \sum_{(a)(x)} a''^{x''} \otimes a' \cdot x'$$

for all $a, b \in A$ *and* $x, y \in X$.

Theorem

Let (X, A) be a matched pair of bialgebras. There exists a unique bialgebra structure on the vector space $X \otimes A$, called the bicrossed product of X and A and denoted by $X \bowtie A$, such that its product, unit, coproduct and counit are given by

- $(x \otimes a)(y \otimes b) = \sum_{(a)(y)} x(a' \cdot y') \otimes a''^{y''} b,$
- $\eta(1) = 1 \otimes 1,$
- $\Delta(x \otimes a) = \sum_{(a)(x)} (x' \otimes a') \otimes (x'' \otimes a''),$
- $\varepsilon(x \otimes a) = \varepsilon(x)\varepsilon(a)$

  for all $x, y \in X$ and $a, b \in A$.

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- $\varepsilon(x \otimes a) = \varepsilon(x)\varepsilon(a)$
  for all $x, y \in X$ and $a, b \in A$.
- *If the bialgebras X and A have antipodes, $X \bowtie A$ is a Hopf algebra.*

Theorem

Let $H = (H, \mu, \eta, \Delta, \varepsilon, S, S^{-1})$ be a finite-dimensional Hopf algebra and $X = (H^{op})^* = (H^*, \Delta^*, \varepsilon^*, (\mu^{op})^*, \eta^*, (S^{-1})^*, S^*)$ be the dual of the opposite Hopf algebra. Let $\alpha : H \otimes X \to X$ and $\beta : H \otimes X \to H$ be the linear maps given by

$$\alpha(a \otimes f) = a \cdot f = \sum_{(a)} f(S^{-1}(a'')?a'), \quad \textit{and}$$

$$\beta(a \otimes f) = a^f = \sum_{(a)} f(S^{-1}(a''')a')a''$$

for $a \in H$ and $f \in X$, where $f(S^{-1}(a'')?a')$ is the map defined by $f(S^{-1}(a'')?a')(x) = f(S^{-1}(a'')xa')$, for all $x \in H$. Then the pair (H, X) is matched.

Quantum double

Definition

The quantum double of H is defined by

$$D(H) = X \bowtie H$$

where H is a finite-dimensional Hopf algebra with invertible antipode and $X = (H^{op})^*$.

Theorem

Let $\{e_i\}_{i \in I}$ be a basis of H and $\{e^i\}_{i \in I}$ be its dual basis. $D(H)$ is a braided Hopf algebra with the universal R-matrix

$$R = \sum_{i \in I} (1 \otimes e_i) \otimes (e^i \otimes 1).$$

Bialgebra Structure of $\mathrm{M_{p,q}(n)}$

Definition

Let p and q be nonzero elements of a field K and $M_{p,q}(n) = K\{a_{ij} | i, j \in \{1, 2, ..., n\}\}/I$ be the quotient of the free algebra generated by the generators $\{a_{ij} | i, j \in \{1, 2, ..., n\}\}$ over K by the two-sided ideal I generated by the relations

$$
\begin{aligned}
a_{il}a_{ik} &= pa_{ik}a_{il}, \\
a_{jk}a_{ik} &= qa_{ik}a_{jk}, \\
a_{jk}a_{il} &= p^{-1}qa_{il}a_{jk}, \\
a_{jl}a_{ik} &= a_{ik}a_{jl} + (p - q^{-1})a_{jk}a_{il}
\end{aligned}
$$

whenever $j > i$ and $l > k$.

Bialgebra Structure of $\mathrm{M_{p,q}(n)}$

Define coproduct and counit on the generators as follows:

$$\Delta(a_{ij}) = \sum_{k=1}^{n} a_{ik} \otimes a_{kj}$$
$$\varepsilon(a_{ij}) = \delta_{ij}$$

where δ_{ij} is the Kronecker delta and extend these maps to $M_{p,q}(n)$ as algebra maps.

Special Cases

- $p = q \implies M_q(n)$

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- $det_q = 1 \implies SL_q(n)$

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- $det_q = 1 \implies SL_q(n)$
- $det_q \neq 0 \implies GL_q(n)$

Definition

Let $U_q gl(n)$ be the algebra generated by $e_i, f_i, k_j, k_j^{-1}, i = 1, 2, ..., n-1, j = 1, 2, ..., n$ with the following relations:

$$k_i k_j = k_j k_i,$$
$$k_i e_j k_i^{-1} = q^{\delta_{i,j} - \delta_{i,j+1}} e_j,$$
$$k_i f_j k_i^{-1} = q^{-\delta_{i,j} + \delta_{i,j+1}} f_j,$$
$$e_i f_j - f_j e_i = \delta_{i,j} \frac{k_i k_{i+1}^{-1} - k_i^{-1} k_{i+1}}{q - q^{-1}},$$
$$e_i e_j = e_j e_i, f_i f_j = f_j f_i, \text{ if } |i-j| \geq 2,$$
$$e_i^2 e_{i\pm1} + e_{i\pm1} e_i^2 = (q + q^{-1}) e_i e_{i\pm1} e_i,$$
$$f_i^2 f_{i\pm1} + f_{i\pm1} f_i^2 = (q + q^{-1}) f_i f_{i\pm1} f_i.$$

Hopf Algebra Structure of $\mathrm{U_q gl(n)}$

Define coproduct, counit and antipode on the generators as follows:

$$
\begin{aligned}
\Delta(k_i^{\pm 1}) &= k_i^{\pm 1} \otimes k_i^{\pm 1}, \\
\Delta(e_i) &= e_i \otimes k_i k_{i+1}^{-1} + 1 \otimes e_i, \\
\Delta(f_i) &= f_i \otimes 1 + k_i^{-1} k_{i+1} \otimes f_i, \\
\varepsilon(k_i^{\pm 1}) &= 1, \\
\varepsilon(e_i) &= \varepsilon(f_i) = 0, \\
S(k_i) &= k_i^{-1}, \\
S(e_i) &= -e_i k_i^{-1} k_{i+1}, \\
S(f_i) &= -k_i k_{i+1}^{-1} f_i.
\end{aligned}
$$

and extend Δ and ε on $U_q gl(n)$ as algebra homomorphisms and S as an algebra antihomomorphism.

Definition

Let
$R_{p,q}(n) = K\{x_i^{(k)}, y_i^{(k)} | k \in \{1, 2, ..., n-1\}, i \in \{1, 2, ..., 2n-1\}\}/J$
be the quotient of the free algebra over K *generated by the generators* $\{x_i^{(k)}, y_i^{(k)} | k \in \{1, 2, ..., n-1\}, i \in \{1, 2, ..., 2n-1\}\}$ *by the two-sided ideal* J *generated by the relations*

Definition

$$x_{2i}^{(k)}x_{2i-1}^{(k)} = px_{2i-1}^{(k)}x_{2i}^{(k)}, \qquad x_{2i+1}^{(k)}x_{2i}^{(k)} = qx_{2i}^{(k)}x_{2i+1}^{(k)},$$
$$x_i^{(k)}x_j^{(k)} = x_j^{(k)}x_i^{(k)}, \qquad x_i^{(k_1)}x_l^{(k_2)} = x_l^{(k_2)}x_i^{(k_1)},$$
$$y_{2i+1}^{(k)}y_{2i}^{(k)} = py_{2i}^{(k)}y_{2i+1}^{(k)}, \qquad y_{2i}^{(k)}y_{2i-1}^{(k)} = qy_{2i-1}^{(k)}y_{2i}^{(k)},$$
$$y_i^{(k)}y_j^{(k)} = y_j^{(k)}y_i^{(k)}, \qquad y_i^{(k_1)}y_l^{(k_2)} = y_l^{(k_2)}y_i^{(k_1)},$$
$$x_i^{(k_3)}y_l^{(k_4)} = y_l^{(k_4)}x_i^{(k_3)}$$

for every $i, j, k, l, k_1, k_2, k_3, k_4$ *where* $k_1 \neq k_2$, $|j - i| \geq 2$.

$$X^{(k)} = \begin{pmatrix} x_1^{(k)} & x_2^{(k)} & & & \\ & x_3^{(k)} & x_4^{(k)} & & \\ & & \ddots & \ddots & \\ & & & & x_{2n-2}^{(k)} \\ & & & & x_{2n-1}^{(k)} \end{pmatrix}$$

$$Y^{(k)} = \begin{pmatrix} y_1^{(k)} & & & & \\ y_2^{(k)} & y_3^{(k)} & & & \\ & y_4^{(k)} & y_5^{(k)} & & \\ & & \ddots & \ddots & \\ & & & y_{2n-2}^{(k)} & y_{2n-1}^{(k)} \end{pmatrix}$$

Theorem

The map $\phi : M_{p,q}(n) \to R_{p,q}(n)$ *mapping* a_{ij} *to* $\hat{a}_{ij}$*, where* $\hat{a}_{ij}$ *is the* ij*th entry of the matrix* $\hat{A} = X^{(1)}X^{(2)}...X^{(n-1)}Y^{(1)}Y^{(2)}...Y^{(n-1)}$*, is well-defined, i.e. the entries of* $\hat{A} = (\hat{a}_{ij})$ *satisfy relations*

$$\begin{aligned}
\hat{a}_{il}\hat{a}_{ik} &= p\hat{a}_{ik}\hat{a}_{il}, \\
\hat{a}_{jk}\hat{a}_{ik} &= q\hat{a}_{ik}\hat{a}_{jk}, \\
\hat{a}_{jk}\hat{a}_{il} &= p^{-1}q\hat{a}_{il}\hat{a}_{jk}, \\
\hat{a}_{jl}\hat{a}_{ik} &= \hat{a}_{ik}\hat{a}_{jl} + (p - q^{-1})\hat{a}_{jk}\hat{a}_{il}
\end{aligned}$$

whenever $j > i$ *and* $l > k$.

Sketch of the Proof

Use infinite matrices $\tilde{X}^{(1)}, \tilde{X}^{(2)}, ..., \tilde{X}^{(n)}$.

Lemma

$$(\tilde{X}^{(1)}\tilde{X}^{(2)}...\tilde{X}^{(n)})_{ij} = \begin{cases} x^{(1)}_{2i-1}x^{(2)}_{2i-1}...x^{(n)}_{2i-1} & \mathit{if}\ i=j \\ \sum_{k_{j-i}=j-i}^{n} ... \sum_{k_2=2}^{k_3-1}\sum_{k_1=1}^{k_2-1}\omega & \mathit{if}\ 0<j-i\leq n \\ 0 & \mathit{otherwise} \end{cases}$$

where $\omega = x^{(1)}_{2i-1}x^{(2)}_{2i-1}...x^{(k_1-1)}_{2i-1}x^{(k_1)}_{2i}x^{(k_1+1)}_{2i+1}...$
$x^{(k_{j-i}-1)}_{2j-3}x^{(k_{j-i})}_{2(j-1)}x^{(k_{j-i}+1)}_{2j-1}...x^{(n-1)}_{2j-1}x^{(n)}_{2j-1}.$

$$\tilde{A}_{ij}^{(n)} = \tilde{A}_{ij-1}^{(n-1)} x_{2j-2}^{(n)} + \tilde{A}_{ij}^{(n-1)} x_{2j-1}^{(n)},$$

If $d > c$

$$\begin{aligned}
\tilde{A}_{ad}^{(n)} \tilde{A}_{ac}^{(n)} &= (\tilde{A}_{ad-1}^{(n-1)} x_{2d-2}^{(n)} + \tilde{A}_{ad}^{(n-1)} x_{2d-1}^{(n)})(\tilde{A}_{ac-1}^{(n-1)} x_{2c-2}^{(n)} + \tilde{A}_{ac}^{(n-1)} x_{2c-1}^{(n)}) \\
&= \tilde{A}_{ad-1}^{(n-1)} x_{2d-2}^{(n)} \tilde{A}_{ac-1}^{(n-1)} x_{2c-2}^{(n)} + \tilde{A}_{ad}^{(n-1)} x_{2d-1}^{(n)} \tilde{A}_{ac-1}^{(n-1)} x_{2c-2}^{(n)} \\
&+ \tilde{A}_{ad-1}^{(n-1)} x_{2d-2}^{(n)} \tilde{A}_{ac}^{(n-1)} x_{2c-1}^{(n)} + \tilde{A}_{ad}^{(n-1)} x_{2d-1}^{(n)} \tilde{A}_{ac}^{(n-1)} x_{2c-1}^{(n)} \\
&= p\tilde{A}_{ac-1}^{(n-1)} x_{2c-2}^{(n)} \tilde{A}_{ad-1}^{(n-1)} x_{2d-2}^{(n)} + p\tilde{A}_{ac-1}^{(n-1)} x_{2c-2}^{(n)} \tilde{A}_{ad}^{(n-1)} x_{2d-1}^{(n)} \\
&+ p\tilde{A}_{ac}^{(n-1)} x_{2c-1}^{(n)} \tilde{A}_{ad-1}^{(n-1)} x_{2d-2}^{(n)} + p\tilde{A}_{ac}^{(n-1)} x_{2c-1}^{(n)} \tilde{A}_{ad}^{(n-1)} x_{2d-1}^{(n)} \\
&= p(\tilde{A}_{ac-1}^{(n-1)} x_{2c-2}^{(n)} + \tilde{A}_{ac}^{(n-1)} x_{2c-1}^{(n)})(\tilde{A}_{ad-1}^{(n-1)} x_{2d-2}^{(n)} + \tilde{A}_{ad}^{(n-1)} x_{2d-1}^{(n)}) \\
&= p\tilde{A}_{ac}^{(n)} \tilde{A}_{ad}^{(n)}
\end{aligned}$$

We will follow the method of Marc Rosso in
An Analogue of P.B.W. Theorem and the Universal R-Matrix for $U_h sl(N+1)$

Let $\alpha = \alpha(i, j+1) = \alpha_i + \alpha_{i+1} + ... + \alpha_j$, $\gamma = \alpha - \alpha_j$ and $\beta = \alpha - \alpha_i$ where $i \neq j$. Then define by induction

$$e_\alpha = \begin{cases} e_\gamma e_j - q e_j e_\gamma & \text{if } i \neq j \\ e_i & \text{if } i = j \end{cases}$$

$$f_\alpha = \begin{cases} f_i f_\beta - q^{-1} f_\beta f_i & \text{if } i \neq j \\ f_i & \text{if } i = j \end{cases}$$

We order the elements as follows:

$$e_{\alpha(i,j)} < e_{\alpha(k,l)} \text{ if } i > k \text{ or } (i = k \text{ and } j > l)$$
$$f_{\alpha(i,j)} < f_{\alpha(k,l)} \text{ if } i < k \text{ or } (i = k \text{ and } j < l)$$

Proposition

Let $\alpha = \alpha_i + ... + \alpha_j$ *and* $\beta = \alpha_p + ... + \alpha_r$. *Upto exchanging the roles of* α *and* β *we may assume* $i \leq p$. *Then,*

$$e_\alpha e_\beta = \begin{cases} e_\beta e_\alpha & \textit{if} \quad p \geq j+2 \\ q e_\beta e_\alpha + e_{\alpha+\beta} & \textit{if} \quad p = j+1 \\ q^{-1} e_\beta e_\alpha & \textit{if} \quad p = i \ \textit{and} \ r \geq j+1 \\ e_\beta e_\alpha & \textit{if} \quad i < p < j \ \textit{and} \ r < j \\ q^{-1} e_\beta e_\alpha & \textit{if} \quad i < p \leq j \ \textit{and} \ r = j \\ e_\beta e_\alpha - (q - q^{-1}) e_{\alpha'} e_{\alpha''} & \textit{if} \quad i < p \leq j \ \textit{and} \ r \geq j+1 \end{cases}$$

where $\alpha' = \alpha_p + ... + \alpha_j$ *and* $\alpha'' = \alpha_i + ... + \alpha_r$.

Proposition

Let $\alpha = \alpha_i + \ldots + \alpha_j$ and $\beta = \alpha_p + \ldots + \alpha_r$. Upto exchanging the roles of α and β we may assume $i \leq p$. Then,

$$f_\beta f_\alpha = \begin{cases} f_\alpha f_\beta & \textit{if} \quad p \geq j+2 \\ q(f_\alpha f_\beta - f_{\alpha+\beta}) & \textit{if} \quad p = j+1 \\ q^{-1} f_\alpha f_\beta & \textit{if} \quad p = i \textit{ and } r \geq j+1 \\ f_\alpha f_\beta & \textit{if} \quad i < p < j \textit{ and } r < j \\ q^{-1} f_\alpha f_\beta & \textit{if} \quad i < p \leq j \textit{ and } r = j \\ f_\alpha f_\beta - (q - q^{-1}) f_{\alpha'} f_{\alpha''} & \textit{if} \quad i < p \leq j \textit{ and } r \geq j+1 \end{cases}$$

where $\alpha' = \alpha_i + \ldots + \alpha_r$ and $\alpha'' = \alpha_p + \ldots + \alpha_j$.

Theorem

1. *The set* $B^0 = \{\prod_i^n k_i^{c_i} : c_i \in \mathbb{Z}\}$ *is a basis for* $U_q^0 gl(n)$.

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2. *The set*
$$B^+ = \{\prod_{\alpha\in\Phi^+} e_\alpha^{c_\alpha} : c_\alpha \in \mathbb{N}\},$$
where the product is in the order corresponding to that of the elements e_α, *is a basis for* $U_q^+ gl(n)$.

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$$B^- = \{\prod_{\alpha\in\Phi^+} f_\alpha^{c_\alpha} : c_\alpha \in \mathbb{N}\},$$
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Theorem

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3. *The set*
$$B^- = \{\prod_{\alpha \in \Phi^+} f_\alpha^{c_\alpha} : c_\alpha \in \mathbb{N}\},$$
where the product is in the order corresponding to that of the elements f_α, *is a basis for* $U_q^- gl(n)$.
4. *Hence the set* $B = B^- \otimes B^0 \otimes B^+$ *is a basis for* $U_q gl(n)$.

Definition

Let $(U, \mu, \eta, \Delta, \varepsilon)$ and $(H, \mu, \eta, \Delta, \varepsilon)$ be bialgebras and $<,>$ be a bilinear form on $U \times H$. We say that the bilinear form realizes a duality between U and H, or that the bialgebras U and H are in duality if we have

$$< uv, x > = \sum_{(x)} < u, x' >< v, x'' >, \tag{1}$$

$$< u, xy > = \sum_{(u)} < u', x >< u'', y >, \tag{2}$$

$$< 1, x > = \varepsilon(x), \tag{3}$$

$$< u, 1 > = \varepsilon(u) \tag{4}$$

for all $u, v \in U$ and $x, y \in H$.

Definition

Moreover, if U and H are Hopf algebras with antipode S, then they are said to be in duality if the underlying bialgebras are in duality and we have

$$< S(u), x > = < u, S(x) >$$

for all $u \in U$ and $x \in H$.

Proposition

Let ϕ be the linear map from U to the dual vector space H^* and ψ be the linear map from H to the dual vector space U^* defined by

$$\phi(u)(x) =< u, x > \qquad \psi(x)(u) =< u, x >$$

With the above notation, the relations (1) and (3) of the previous definition are equivalent to ϕ being an algebra morphism and the relations (2) and (4) are equivalent to ψ being an algebra morphism.

Construct an algebra map ψ from $M_q(n)$ to the dual algebra $U_q^* gl(n)$.
Consider the representation ρ defined on the generators by

$$\rho(e_i) = E_{i,i+1},$$
$$\rho(f_i) = E_{i+1,i},$$
$$\rho(k_i) = D_i,$$

where E_{ij} denotes the elementary matrix and D_i denotes the diagonal matrix

$$D_i = E_{11} + E_{22} + ... + qE_{i,i} + E_{i+1,i+1} + E_{i+2,i+2} + ... + E_{nn}.$$

If u is an element of $U_q gl(n)$ using the P.B.W. basis, we have

$$\rho(u) = \begin{pmatrix} A_{11}(u) & A_{12}(u) & \dots & A_{1n}(u) \\ A_{21}(u) & A_{22}(u) & \dots & A_{2n}(u) \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1}(u) & A_{n2}(u) & \dots & A_{nn}(u) \end{pmatrix}$$

Let $\psi : H = M_q(n) \to U^* = U_q gl(n)^*$ be the algebra morphism defined on the generators by $\psi(a_{ij}) = A_{ij}$.

Theorem

The bilinear form $< u, x >= \psi(x)(u)$ realizes a duality between the bialgebras $U_q gl(n)$ and $M_q(n)$.

- ψ is well-defined.

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- $<,>$ satisfies (1) and (3)

Thank You ☺