The invariant subspace problem via compact-friendly-like operators: a survey

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Outline

Fix a Banach space X and $T \in \mathcal{L}(X)$. A subspace V of X is called **non-trivial** if $\{0\} \neq V \neq X$. If $TV \subseteq V$, then V is called a T**-invariant subspace**. If V is S-invariant for every S in the commutant

$$\{T\}' := \{S \in \mathcal{L}(X) \mid ST = TS\}$$

of T, then V is called a T**-hyperinvariant subspace**.

The Invariant Subspace Problem

When does a bounded operator on a Banach space have a non-trivial closed invariant subspace?

An operator $T : X \to Y$ between normed spaces is called **compact** if $\overline{TX_1}$ is a compact set in Y, where X_1 is the closed unit ball of X. The family of all compact operators from X to Y is denoted by $\mathcal{K}(X, Y)$, and one defines $\mathcal{K}(X) := \mathcal{K}(X, X)$.

An operator $T \in \mathcal{L}(X)$ on a Banach space X is said to be:

- **quasi-nilpotent** if $r(T) = \lim_{n\to\infty} \|T^n\|^{1/n} = 0$,
- **locally quasi-nilpotent** at a vector x in X if $r_T(x) := \lim_{n\to\infty} \|T^n x\|^{1/n} = 0$,
- **essentially quasi-nilpotent** if $r_{ess}(T) := r(\pi(T)) = 0$, where $\pi : \mathcal{L}(X) \to \mathfrak{C}(X)$ is the quotient map of $\mathcal{L}(X)$ onto the Calkin algebra $\mathfrak{C}(X) := \mathcal{L}(X)/\mathcal{K}(X)$.

A **non-scalar** operator is one which is not a multiple of the identity.

Theorem (V. Lomonosov – 1973)

If an operator $T : X \to X$ on a complex Banach space commutes with a non-scalar operator $S \in \mathcal{L}(X)$ which in turn commutes with a non-zero compact operator, then T has a non-trivial closed invariant subspace.

Corollary

If a non-scalar operator T on a complex Banach space commutes with a non-zero compact operator, then T has a non-trivial closed hyperinvariant subspace.

An operator $T \in \mathcal{L}(X)$ is a **Lomonosov operator** if there is an operator $S \in \mathcal{L}(X)$ such that:

- S is a non-scalar operator.
- S commutes with T.
- S commutes with a non-zero compact operator.

Lomonosov's Theorem redux

Every Lomonosov operator on a complex Banach space has a non-trivial closed invariant subspace.

But not vice versa: not every operator with a non-trivial closed invariant subspace is a Lomonosov operator (D.W. Hadwin, E.A. Nordgren, H. Radjavi & P. Rosenthal—1980).

- A **Banach lattice** is a real Banach space E equipped with a partial order $\leqslant$, which makes E into a lattice in the algebraic sense and which is compatible with the linear and the norm structures.
- Complex Banach lattices are obtained via complexification of real ones.
- All classical Banach spaces are Banach lattices under their natural orderings and norms.
- For a Banach lattice E, the set

$$E^+ := \{x \in E \mid x \geqslant 0\}$$

is referred to as the **(positive) cone** of E.

- The infimum and the supremum operations $\wedge$ and $\vee$ in a Banach lattice E generate the positive elements

$$x^+ := x \vee 0, \quad x^- := (-x) \vee 0, \quad |x| := x \vee (-x),$$

for which the identities

$$x = x^+ - x^-, \quad |x| = x^+ + x^-, \quad x^+ \wedge x^- = 0$$

hold for every x in E.
- If V is a subspace of a Banach lattice and if $v \in V$ and $|u| \leqslant |v|$ imply $u \in V$, then V is called an **ideal**.

- A linear operator $T : E \to F$ between Banach lattices is called **positive** and is denoted by $T \geqslant 0$ if
  $TE^+ \subseteq F^+$.
  The family of positive operators from E to F is denoted
  by $\mathcal{L}(E, F)_+$.
- The family $\mathcal{L}(E, F)$ becomes an ordered vector space with $\mathcal{L}(E, F)_+$ being its positive cone by declaring

$$T \geqslant S \quad \text{whenever} \quad T - S \geqslant 0.$$

- An operator T on E is said to be **dominated** by a positive operator B on E, denoted by $T \prec B$, provided

$$|Tx| \leqslant B|x|$$

  for each $x \in E$.

- An operator on E which is dominated by a multiple of the identity operator is called a **central operator.** The collection of all central operators on E is denoted by $\mathcal{Z}(E)$ and is referred to as the **center** of the Banach lattice E.
- An operator $T : E \to F$ is said to be *AM*-**compact**, provided that T maps order bounded sets to norm-precompact sets. Each compact operator is necessarily *AM*-compact.

The ISP for positive operators on Banach lattices

Does every positive operator on an infinite-dimensional, separable Banach lattice have a non-trivial closed invariant subspace?

Notation: Throughout, the letters X and Y will denote infinite-dimensional Banach spaces while E and F will be fixed infinite-dimensional Banach lattices.

Compact-friendliness (Y.A. Abramovich, C.D. Aliprantis & O. Burkinshaw–1994)

A positive operator B on E is said to be **compact-friendly** if there is a positive operator that commutes with B and dominates a non-zero operator which is dominated by a compact positive operator.

In other words, a positive operator B on E is compact-friendly if there exist three non-zero operators $R, K, C : E \to E$ such that R, K are positive, K is compact, $RB = BR$, and for each $x \in E$ one has

$$|Cx| \leqslant R|x| \quad \text{and} \quad |Cx| \leqslant K|x|.$$

Some examples of compact-friendly operators are:

- Compact positive operators: if $B \geqslant 0$ is compact, then take $R = K = C := B$ in the definition.
- The identity operator: having fixed an arbitrary non-zero compact positive operator K, set $R = C := K$ (which also shows that a compact-friendly operator need not be compact).
- Every power (even every polynomial with non-negative coefficients) of a compact-friendly operator.
- Positive operators that commute with a non-zero positive compact operator.
- Positive operators that dominate or that are dominated by non-zero positive compact operators.
- Positive integral operators.

A continuous function $\varphi : \Omega \to \mathbb{R}$, where Ω is a topological space, has a **flat** if there exists a non-empty open set Ω_0 in Ω such that φ is constant on Ω_0.

If Ω is a compact Hausdorff space, then each $\varphi \in \mathcal{C}(\Omega)$ generates the **multiplication operator** $M_\varphi : \mathcal{C}(\Omega) \to \mathcal{C}(\Omega)$ defined for each $f \in \mathcal{C}(\Omega)$ by

$$M_\varphi f = \varphi f.$$

The function φ is called the **multiplier** of M_φ.

A multiplication operator M_φ is positive if and only if the multiplier φ is positive.

Theorem (Y.A. Abramovich, C.D. Aliprantis & O. Burkinshaw−1997)

A positive multiplication operator M_φ on a $C(\Omega)$-space, where Ω is a compact Hausdorff space, is compact-friendly if and only if the multiplier φ has a flat.

Apart from its counterparts, this is the only known characterization of compact-friendliness on a concrete Banach lattice. A similar characterization of compact-friendly multiplication operators on L_p-spaces was obtained by Y.A. Abramovich, C.D. Aliprantis, O. Burkinshaw and A.W. Wickstead in 1998. G. Sirotkin has managed to extend the latter to arbitrary Banach function spaces in 2002.

Theorem (Y.A. Abramovich, C.D. Aliprantis & O. Burkinshaw–1998)

If a non-zero compact-friendly operator $B : E \to E$ on a Banach lattice E is quasi-nilpotent at some $x_0 > 0$, then B has a non-trivial closed invariant ideal.

The motivation

Let B and R be two commuting positive operators on E such that B is compact-friendly and R is locally quasi-nilpotent at some non-zero positive vector in E. Does there exist a non-trivial closed B-invariant subspace, or an R-invariant subspace, or a common invariant subspace for B and R?

Quasi-similarity (B. Sz.-Nagy & C. Foiaş–1970)

- An operator $Q \in \mathcal{L}(X, Y)$ is a **quasi-affinity** if Q is one-to-one and has dense range.
- An operator $T \in \mathcal{L}(X)$ is said to be a **quasi-affine transform** of an operator $S \in \mathcal{L}(Y)$ if there exists a quasi-affinity $Q \in \mathcal{L}(X, Y)$ such that $QT = SQ$.
- If both T and S are quasi-affine transforms of each other, the operators $T \in \mathcal{L}(X)$ and $S \in \mathcal{L}(Y)$ are called **quasi-similar** and this is denoted by $T \overset{qs}{\sim} S$.

- Similarity implies quasi-similarity; the implication is generally not reversible.
- Quasi-similarity is an equivalence relation on the class of all operators.
- Quasi-similarity and commutativity are different notions: neither of them implies the other.

Positive quasi-similarity

Two positive operators $S \in \mathcal{L}(E)$ and $T \in \mathcal{L}(F)$ are **positively quasi-similar**, denoted by $S \overset{pqs}{\sim} T$, if there exist positive quasi-affinities $P \in \mathcal{L}(E, F)$ and $Q \in \mathcal{L}(F, E)$ such that $TP = PS$ and $QT = SQ$.

- Since positive operators between Banach lattices are continuous, each operator dominated by a positive operator is automatically continuous. This guarantees the non-triviality of an operator of the form QTP if P and Q are positive quasi-affinities on E, whenever T is a non-trivial operator dominated by a positive operator on E.

Lemma

Let B and T be two positive operators on E. If B is compact-friendly and T is positively quasi-similar to B, then T is also compact-friendly.

Sketch of proof.

- Since $T \overset{pqs}{\sim} B$, there exist quasi-affinities P and Q such that $BP = PT$ and $QB = TQ$.
- As B is compact-friendly, there exist three non-zero operators R, K, and C on E with R, K positive and K compact such that $RB = BR$, $C \prec R$, and $C \prec K$.
- Take $R_1 := QRP$, $K_1 := QKP$, and $C_1 := QCP$ as the required three operators for the compact-friendliness of T.

□

- Although quasi-similarity need not preserve compactness (T.B. Hoover$-$1972), positive quasi-similarity does preserve compact-friendliness.
- There exists a non-zero quasi-nilpotent operator on ℓ_2 that does not commute with any non-zero compact operator, and hence is not quasi-similar to any compact operator (C. Foiaş & C. Pearcy$-$1974). Combined with a result of H.H. Schaefer which dates back to 1970, an example in the same spirit for Banach lattices is obtained: *there exists a positive quasi-nilpotent operator on the Banach lattice $L_p(\mu)$, where $1 \leqslant p < \infty$ and μ is the Lebesgue measure on the unit circle $\mathbb{T}$, which is not positively quasi-similar to any non-zero compact-friendly operator*.

Theorem

Let B and T be two positive operators on E such that B is compact-friendly and T is locally quasi-nilpotent at a non-zero positive element of E. If $B \overset{pqs}{\sim} T$, then T has a non-trivial closed invariant ideal.

Strongly compact-friendly operators

A positive operator B on a Banach lattice E is called **strongly compact-friendly** if there exist three non-zero operators R, K, and C on E with R, K positive, K compact such that $B \overset{pqs}{\sim} R$, and C is dominated by both R and K.

Denote the families of positive compact operators, strongly compact-friendly operators and compact-friendly operators
on E by $\mathcal{K}(E)$, $\mathcal{SKF}(E)$ and $\mathcal{KF}(E)$, respectively

Lemma

(i) *If a positive operator B on E is positively quasi-similar to an operator on E which is dominated by a positive compact operator or which dominates a positive compact operator, then B is strongly compact-friendly, and the commutant $\{B\}'$ of B contains an operator which is dominated by a positive compact operator or which dominates a positive compact operator, respectively. In particular, every positive operator which is positively quasi-similar to a positive compact operator is strongly compact-friendly and commutes with a positive compact operator.*

(ii) *A non-zero positive operator B on E is strongly compact-friendly if and only if λB is strongly compact-friendly for some scalar $\lambda > 0$. However, B need not be quasi-similar to λB for $\lambda \neq 1$.*

(Continued)

(iii) *A positive compact perturbation of a positive operator on E is strongly compact-friendly.*
(iv) *For every positive operator B on E, there exists a strongly compact-friendly operator T on E which dominates B.*
(v) *If $B \geqslant I$ on E and $\{B\}'$ does not contain a non-zero compact operator, then there exists a non-zero strongly compact-friendly, non-compact operator on E which is not positively quasi-similar to B.*
(vi) *Positive kernel operators on order-complete Banach lattices are strongly compact-friendly.*
(vii) *Every non-zero positive operator on ℓ_p $(1 \leqslant p < \infty)$ is strongly compact-friendly.*

Theorem

For an infinite-dimensional Banach lattice E, one has

$$\mathcal{K}(E)_+ \subset \mathcal{SKF}(E) \subset \mathcal{KF}(E)$$

and the inclusions are generally proper.

Sketch of proof.

- That both inclusions hold and that the former is proper follow from (i) and (iii) of the previous Lemma.
- For a compact Hausdorff space Ω without isolated points, the space $E := \mathcal{C}(\Omega)$ and the identity operator on E provide together an example which reveals that the second inclusion may well be proper.

□

Example. ***A strongly compact-friendly operator which is not polynomially compact***

Let $T : \ell_2 \to \ell_2$ be the backward weighted shift defined by

$$Te_0 = 0 \quad \text{and} \quad Te_{n+1} = \tau_n e_n, \quad n \geqslant 0,$$

where $(e_n)_{n=0}^{\infty}$ is the canonical basis of ℓ_2 and $(\tau_n)_{n=0}^{\infty}$ is the sequence

$$\left(\frac{1}{2}, \frac{1}{2^4}, \frac{1}{2}, \frac{1}{2^{16}}, \frac{1}{2}, \frac{1}{2^4}, \frac{1}{2}, \frac{1}{2^{64}}, \frac{1}{2}, \frac{1}{2^4}, \frac{1}{2}, \frac{1}{2^{256}}, \dots\right).$$

(C. Foiaş & C. Pearcy—1974)

One can observe that:

- T is a positive, non-compact operator.
- T is strongly compact-friendly.
- $\|T^n\|^{1/n} \to 0$, that is, T is quasi-nilpotent, and hence is essentially quasi-nilpotent.
- No power of T is compact.

It then follows that T is not polynomially compact.

On a subclass of $\mathcal{SKF}(E)$, the local quasi-nilpotence assumption in (Y.A. Abramovich, C.D. Aliprantis & O. Burkinshaw$-$1998) can be removed.

Theorem

Let B be a positive operator on E. If B is positively quasi-similar to a positive operator R on E which is dominated by a positive compact operator K on E, then B has a non-trivial closed invariant subspace. Moreover, for each sequence $(T_n)_{n\in\mathbb{N}}$ in $\{B\}'$, there exists a non-trivial closed subspace that is invariant under B and under each T_n.

For a positive operator B on a Banach lattice E, the **super right-commutant** $[B\rangle$ of B is defined by

$$[B\rangle := \{A \in \mathcal{L}(E)_+ \mid AB - BA \geqslant 0\}.$$

A subspace of E which is A-invariant for every operator A in $[B\rangle$ is called a $[B\rangle$**-invariant subspace**.

Weakly compact-friendly operators

A positive operator $B \in \mathcal{L}(E)$ is called **weakly compact-friendly** if there exist three non-zero operators R, K, and C on E with R, K positive and K compact such that $R \in [B\rangle$, and C is dominated by both R and K.

Weak positive quasi-similarity

Two positive operators $B \in \mathcal{L}(E)$ and $T \in \mathcal{L}(F)$ are **weakly positively quasi-similar**, denoted by $B \overset{w}{\sim} T$, if there exist positive quasi-affinities $P \in \mathcal{L}(F, E)$ and $Q \in \mathcal{L}(E, F)$ such that $BP \leqslant PT$ and $TQ \leqslant QB$.

Theorem

The binary relation $\overset{w}{\sim}$ is an equivalence relation on the class of all positive operators, under which weak compact-friendliness is preserved.

Example. ***A weakly compact-friendly operator which is not compact-friendly***

Let $E := \mathcal{C}[0,1/2]$, equipped with the uniform norm. Define $\varphi : [0,1/2] \to \mathbb{R}$ by $\varphi(\omega) := 1 - 2\omega$ for all $\omega \in [0,1/2]$. The multiplication operator $M_\varphi : E \to E$ is not compact-friendly since φ has no flats. But M_φ is weakly compact-friendly: take the linear functional $\psi \in E^*$ given by $\psi(f) := f(0)$ for all $f \in E$ and define the rank-one (and hence, compact) positive operator $K : E \to E$ by

$$Kf := (\psi \otimes \varphi)(f), \quad f \in E.$$

Set $R = C := K$.

The arguments, with slight modifications, used in the proof of the corresponding theorem of Abramovich, Aliprantis and Burkinshaw concerning the commutant of the operator B can be shown to work for weakly compact-friendly operators as well.

Theorem

If a non-zero weakly compact-friendly operator $B : E \to E$ on a Banach lattice is quasi-nilpotent at some $x_0 > 0$, then B has a non-trivial closed invariant ideal. Moreover, for each sequence $(T_n)_{n\in\mathbb{N}}$ in $[B\rangle$ there exists a non-trivial closed ideal that is invariant under B and under each T_n.

Theorem

Let T be a locally quasi-nilpotent positive operator which is weakly positively quasi-similar to a compact operator. Then T has a non-trivial closed invariant subspace.

Sketch of proof.

- If $T \overset{w}{\sim} K$ with K compact, then there exist positive quasi-affinities P and Q such that $TP \leqslant PK$ and $KQ \leqslant QT$. Thus, $TPKQ \leqslant PK^2Q \leqslant PKQT$, i.e., the compact operator $K_0 := PKQ$ belongs to $[T\rangle$.
- Being also weakly-compact friendly by the next-to-last Theorem, the locally quasi-nilpotent operator T has a non-trivial closed invariant ideal by the previous Theorem.

□

Topological fullness of the center (A.W. Wickstead–1981)

The center $\mathcal{Z}(E)$ of a Banach lattice E is called **topologically full** if whenever $x, y \in E$ with $0 \leqslant x \leqslant y$, one can find a sequence $(T_n)_{n\in\mathbb{N}}$ in $\mathcal{Z}(E)$ such that $\|T_n y - x\| \to 0$.

Some examples are:

- Banach lattices with quasi-interior points–such as separable Banach lattices.
- Dedekind σ-complete Banach lattices–such as L_p-spaces.

The result in (J. Flores, P. Tradacete & V.G. Troitsky–2008), which uses the existence of a quasi-interior point, can further be improved.

Theorem

Suppose that B is a positive operator on a Banach lattice E with topologically full center such that

(i) *B is locally quasi-nilpotent at some $x_0 > 0$, and*

(ii) *there is an $S \in [B\rangle$ such that S dominates a non-zero AM-compact operator K.*

Then $[B\rangle$ has an invariant closed ideal.

Sketch of proof.

- Since the null ideal N_B of B is $[B\rangle$-invariant, assume that $N_B = \{0\}$.
- Use the topological fullness of $\mathcal{Z}(E)$ to show that there exists an operator M in

$$\mathcal{Z}(E)_{1+} := \{T \in Z(E) \mid 0 \leqslant T \leqslant I\}$$

with $M|Kz| \neq 0$, where $z \in E$ is such that $Kz \neq 0$.
- Put $K_1 := MK$ and observe that $BK_1 \neq 0$, that BK_1 is AM-compact, and that BK_1 is dominated by BS.
- Observe that the semigroup ideal $\mathcal{J}$ in $[B\rangle$ generated by BS is finitely quasi-nilpotent at x_0, whence $\mathcal{J}$ has an invariant closed ideal.

□

Dedekind completeness and compact-friendliness can be relaxed, respectively, to topological fullness of the center and weak compact-friendliness in (Y.A. Abramovich, C.D. Aliprantis & O. Burkinshaw−1998).

Theorem

Let E be a Banach lattice with topologically full center. If B is a locally quasi-nilpotent weakly compact-friendly operator on E, then $[B\rangle$ has a non-trivial closed invariant ideal.

Sketch of proof.

- For each $x > 0$ denote by J_x the ideal generated by the orbit $[B\rangle x$, and suppose that $\overline{J_x} = E$ for each $x > 0$.
- Use the topological fullness of $\mathcal{Z}(E)$ to show that there exists an operator M_1 in

$$\mathcal{Z}(E)_{1+} := \{T \in Z(E) \mid 0 \leqslant T \leqslant I\}$$

  with $M_1|Cx_1| \neq 0$, where $x_1 > 0$ is such that $Cx_1 \neq 0$.
- Put $\pi_1 := M_1 C$ and observe that π_1 is dominated by R and K.
- Repeat the preceding argument twice more to get a non-zero positive operator S in $[B\rangle$ which dominates a compact operator.
- Invoke the previous Theorem to get the assertion.

SOME OPEN PROBLEMS

- Does every positive operator on ℓ_1 have a non-trivial closed invariant subspace?
- Fix a positive operator B on E, and suppose that there exists a non-zero compact operator dominated by B.
  Does it follow that there exists a non-zero compact *positive* operator dominated by B?
- It is not known whether the set $\mathcal{KF}(E)$ is order-dense in $\mathcal{L}(E)$ for an arbitrary Banach lattice E. Is the set $\mathcal{SKF}(E)$ order-dense in $\mathcal{L}(E)$? In other words, does every strictly positive operator dominate some strongly compact-friendly operator?

- Let B and R be two commuting positive operators on E such that B is compact-friendly and R is locally quasi-nilpotent at some non-zero positive vector in E. Does there exist a non-trivial closed B-invariant subspace, or an R-invariant subspace, or a common invariant subspace for B and R?
- Does every strongly compact-friendly operator have a non-trivial closed invariant subspace?

- Let B and R be two commuting positive operators on E such that B is strongly compact-friendly and R is locally quasi-nilpotent at some non-zero positive vector in E. Does there exist a non-trivial closed B-invariant subspace, or an R-invariant subspace, or a common invariant subspace for B and R?

- Y.A. Abramovich & C.D. Aliprantis, *An Invitation to Operator Theory*, Graduate Studies in Mathematics, Vol. 50, American Mathematical Society, Providence, RI, 2002.
- Y.A. Abramovich, C.D. Aliprantis & O. Burkinshaw, "The invariant subspace problem: some recent advances," in *Workshop on Measure Theory and Real Analysis* (italian) (Grado, 1995); *Rend. Inst. Mat. Univ. Trieste* **29** (1998), suppl., 3-79 (1999).
- C.D. Aliprantis & O. Burkinshaw, *Positive Operators*, Springer, The Netherlands, 2006.
- C. Apostol, H. Bercovici, C. Foiaş & C. Pearcy, "Quasiaffine transforms of operators," *Michigan Math. J.* **29** (1982), no. 2, 243-255.

A. Biswas, A. Lambert & S. Petrovic, “Generalized eigenvalues and the Volterra operator,” *Glasg. Math. J.* **44** (2002), no. 3, 521-534.

M. Çağlar & T. Mısırlıoğlu, “Invariant subspaces of weakly compact-friendly operators,” *Turkish J. Math.* **36** (2012), no. 2, 291-295.

M. Çağlar & T. Mısırlıoğlu, “A note on a problem of Abramovich, Aliprantis and Burkinshaw,” *Positivity* **15** (2011), no. 3, 473-480.

M. Çağlar & T. Mısırlıoğlu, “Weakly compact-friendly operators,” *Vladikavkaz Mat. Zh.* **11** (2009), no. 2, 27-30.

J. Flores, P. Tradacete & V.G. Troitsky, “Invariant subspaces of positive strictly singular operators on

Banach lattices," *J. Math. Anal. Appl.* **343** (2008), no. 2, 743-751.

C. Foiaş & C. Pearcy, "A model for quasinilpotent operators," *Michigan Math. J.* **21** (1974), 399-404.

Y.M. Han, S.H. Lee & W.Y. Lee, "On the structure of polynomially compact operators," *Math. Z.* **232** (1999), no. 2, 257-263.

T.B. Hoover, "Quasi-similarity of operators," *Illinois J. Math.* **16** (1972), 678-686.

K.B. Laursen & M.M. Neumann, *An Introduction to Local Spectral Theory*, London Mathematical Society Monographs, New Series 20, Clarendon Press, Oxford, 2000.

- P. Meyer-Nieberg, *Banach Lattices*, Springer-Verlag, Berlin and New York, 1991.
- G. Sirotkin, "A version of the Lomonosov invariant subspace theorem for real Banach spaces," *Indiana Univ. Math. J.* **54** (2005), no. 1, 257-262.
- B. Sz.-Nagy & C. Foiaş, *Harmonic Analysis of Operators on Hilbert Space*, American Elsevier Publishing Company, Inc., NY, 1970.
- A.W. Wickstead, "Extremal structure of cones of operators," *Quart. J. Math. Oxford* **32** (1981), no. 2, 239-253.
- A.W. Wickstead, "Banach lattices with topologically full centre," *Vladikavkaz Mat. Zh.* **11** (2009), no. 2, 50-60.