SOME DUBIOUS PROPERTIES OF GROUPS OF FINITE MORLEY RANK

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INTRODUCTION

The definition

A group of finite RM is a structure G equipped with a definable group law such that

1. **Each definable set in G^{eq} has a Cantor rank, which is finite**
2. **For uniform families of definable sets, the Cantor rank is definable**
3. **In a uniform family of finite definable sets, the number of elements of the sets is bounded.**

Consequences of the definition

1. **The conditions are preserved under elementary extension (and elementary equivalence)**
2. **The Cantor rank is in fact the Morley rank, that is, it is preserved under elementary extension**
3. **The Cantor rank is additive, and equal to the U-rank of Lascar (and to the weight)**

Algebraic groups

Algebraic groups over an algebraically closed field, equipped with the full structure given by the field, are examples of groups of finite Morley rank, the rank being the geometric dimension.

Many well-known properties of algebraic groups can be extended to the general context of groups of finite RM, and the question is whether they are typical as examples ; more precisely whether a simple group of finite RM must be an algebraic group.

A note for non-model theorists

In an algebraically closed field K, a *definable* set is a *constructible* subset of some Cartesian power of K, that is a finite Boolean combination of Zariski closed sets. Its Morley rank is the same thing as the algebraic dimension of its Zariski closure.

<u>Elimination of quantifiers</u> : the projection on K^n of a constructible subset of K^{n+1} is constructible.

<u>Elimination of imaginaries</u> : at the constructible level we can take quotients, the quotient of any constructible set by a constructible equivalence relation being in constructible bijection with a constructible set.

For instance, from a constructible point of view, the projective line, the affine line with a doubled point, and the affine line plus a point, are the same object.

A constructible group is constructibly isomorphic to an algebraic group G; the constructible subgroups of G are Zariski closed, and the connected ones are irreducible.

Four conjectural properties

The answer to the big question seems now very uncertain, and in fact some properties of algebraic groups seem not to hold in general, although we cannot provide counter-examples.

Many of them are sophisticated facts concerning simple groups, but what we like to discuss here are easily formulated properties of a general character. We mention four of them, that we shall meet in our forthcoming study of the generic centralizers.

1. **Is the Morley degree definable ? Can we define the connected component in a uniform family of definable groups ? (see Hrushovski's paper on fusion for the definability of the Morley degree in the case of algebraically closed fields)**

2. We say that X is almost included in Y if the points of X which are not in Y form a set of rank strictly less than RM(X) .

For any formula $\phi(x,\underline{y})$, we denote by $Cl\phi(X)$ the union of X and of the cosets modulo the subgroups defined by formulae of the kind $\phi(x,\underline{a})$ which are almost included in X .

Do X and $Cl\phi(X)$ have the same rank ? (In the algebraic case, $Cl\phi(X)$ is included in the Zariski-closure of X , which has the same dimension)

3. In a connected group, are the centralizers of the generic points of minimal dimension ? (consequence of the Hauptidealsatz in the algebraic case)

4. A Borel subgroup is a maximal connected definable solvable subgroup of G . Are the Borel subgroups conjugate ? (fixed point theorem for the action of a solvable group on a complete variety in the algebraic case)

What happens in a simple algebraic group ?

In a simple algebraic group, the centralizers of the generic points are connected, conjugated and of finite index in their normalizer.

They are conjugated for a very good reason : there is only a finite number of centralizers of elements of the group, up to conjugacy !

In fact, they are the maximal tori : they are divisible commutative groups, and they contain elements of order p for every prime number except possibly one (the characteristic of the field).

SECTION 1

Generic centralizers and conjugacy classes : basic rank computations

When it is question of Morley rank, we speak of *dimension* for a definable set, RM(X) = dim(X) , and of *rank* for a type, RM(tp(a/A)) = rg(a/A) .

The dimension is not sensitive to the parameters, provided they allow to define the set ; the rank is, since rg(a/A) is the minimal dimension of a set definable over A to which a belongs.

Additivity

For Morley rank : **If f is a definable surjection from X onto Y, each fiber of it being of dimension d, then $\dim(X) = \dim(Y) + d$.**

For Lascar rank : **$rg(a\hat{}b) = rg(a) + rg(b/a)$.**

In general, people prefers the first version ; but we shall use both.

Notations

We consider a group G of finite RM, and a point g of this group, which is <u>generic</u> over $\varnothing$, or any fixed set A of parameters ; rg(g) = dim(G) .

We note C the conjugacy class of g , and c its canonical parameter ; we note Z the centralizer of g , and z its canonical parameter, and we note N the normalizer of Z .

Since Z and its centre Z(Z) are each the centralizer of the other, they have the same canonical parameter, and the same normalizer.

The action of G on itself by inner automorphisms induces a definable (with g as a parameter) bijection between C and the right quotient of G by Z ; all the fibers of this quotient having the same dimension as Z , by additivity :

$$\mathrm{dim}(G) = \mathrm{dim}(Z) + \mathrm{dim}(C) .$$

To compute the rank of c , we need a small lemma.

Lemma 1. ***We consider a definable (over*** $\varnothing$ ***, or*** **A** ***) set*** **X** **,** **g** ***a point of*** **X** ***of maximal rank (i.e.*** **rg(g) = dim(X)** ***),*** **E(x,y)** ***a definable (id.) equivalence relation on*** **X** ***; we note*** **d** ***the dimension of the class*** **E(x,g)** ***and*** **c** ***its canonical parameter. Then*** **rg(c) = dim(X) - d ,** **rg(g/c) = d** ***; in other words*** **g** ***has maximal rank in its class, and, if all the classes have dimension*** **d ,** **c** ***has maximal rank in*** **X/E .**

Therefore :

rg(g/c) = dim(C) , rg(c) = dim(G) - dim(C) = dim(Z) .

We obtain also :

$$\mathrm{rg}(g/z^c) = \dim(N) - \dim(Z)\ .$$

For that, consider the intersection Γ of C and of the centre of Z ; Γ is in bijection with the quotient of N by Z , so that $\dim(\Gamma) = \dim(N) - \dim(Z)$. Then apply the lemma over $\{c\}$, since Γ is the class of g modulo the equivalence relation “to have the same centralizer” (restricted to C).

For the rank of z, we obtain only an inequality. Since g is in Z, and rg(g) = rg(g^z) = rg(z) + rg(g/z) :

$$\mathbf{rg(g/z) \leq dim(Z)\ , \quad rg(z) \geq dim(C)\ .}$$

In fact, g belongs to the center of Z, so that rg(g/z) ≤ dim(Z(Z)) ; when rg(g/z) = dim(Z), Z° is central in Z .

Our ultimate goal is to describe the circumstances when z and c are independent : in a connected group, we shall see that this means that the centralizers of the generic points are conjugated. We begin with a direct consequence of the inequalities above :

Corollary 2. ***If* z *and* c *are independent, then* rg(z) = dim(C) , rg(g/z) = dim(Z) , *and* g *is algebraic over* z^c *(and reciprocally !).***

Some examples of (algebraic) groups

1. If G is commutative, $rg(z) = 0$, $rg(g) = rg(c)$.

2. If G is connected, nilpotent and non-commutative, every proper definable subgroup has infinite index in its normalizer, so that z and c are dependent.

3. In a simple algebraic group G , z and c are independent : the centralizers of the generic points are connected, commutative, and conjugate. In fact, there is only a finite number of centralizers of elements of G up to conjugacy.

4. GL2(K) ; $g = [\alpha\ \beta\ ;\ \gamma\ \delta]$ where α, β, γ et δ are transcendental and independent, rg(g) = 4 ; the conjugacy class C is determined by the set $\{\lambda,\mu\}$ of the eigenvalues, that is the coefficıents of the characteristic polynomial, $c = (\alpha+\beta, \alpha\delta-\beta\gamma)$, rg(c) = 2 = dim(Z) , dim(C) = 2 ; the centralizer is given by the eigenvektors, which form two lines in generic position that can be chosen independently from the eigenvalues, rg(z) = 2 and z is independent from c .

More precisely, the centralizer Z is defined by the system $\gamma.v = \beta.w$ and $\gamma.(u\text{-}t) = (\alpha-\delta).w$, whose canonical coefficients are $z = (\beta/\gamma, \alpha-\delta/\gamma)$.

5. **TL2(K) ; $g = [\alpha\ \beta\ ;\ 0\ \gamma]$, rg(g) = 3 ; C is given by the diagonal, $c = (\alpha,\gamma)$, rg(c) = 2 = dim(Z) , dim(C) = 1 ; Z is defined by the equation $\beta.u + (\alpha-\gamma).v + \beta.w = 0$, with canonical coefficient $\alpha-\gamma/\beta$; rg(z) = 1 , z and c are independent.**

6. **TU3(K) ; $\gamma = [1\ \alpha\ \beta\ ;\ 0\ 1\ \gamma\ ;\ 0\ 0\ 1]$, rg(g) = 3 ; $c = (\alpha,\gamma)$, rg(c) = 2 , dim(Z) = 2 , dim(C) = 1 ; Z is defined by the equation $\alpha.w = \gamma.u$, $z = \alpha/\gamma$ and rg(z) = 1 , but rg(z/c) = 0 , z and c are not independent.**

In a 2-nilpotent group, z is always definable over c !

7. In characteristic p , in $G = [1\ u\ v\ ;\ 0\ 1\ u^p\ ;\ 0\ 0\ 1]$ the generic centralizers are not connected, and in $G = [1\ u\ v\ w\ ;\ 0\ 1\ 0\ t\ ;\ 0\ 0\ 1\ u^p\ ;\ 0\ 0\ 0\ 1]$ the generic centralizers are connected but not commutative.

It is possible to build in characteristic zero a unipotent algebraic group with non commutative (connected !) generic centralizers.

Examples wanted
(if possible algebraic)

Z° commutative but not central in Z .

Z of finite index in N but Z° not central in Z .

SECTION 2

When the generic is generic in its own centralizer ?

We study here the situatıon where rg(g/z) = dim(Z) , as it is the case when z et c are independent.

Theorem 3. ***In a <u>connected</u> group* G *of finite RM tfcae :***

***(i) Each generic point* g *is generic in its own centralizer* Z .**

***(ii)* rg(z) = dim(C) .**

***(iii) The points of* Z *whose centralizer is* Z *form a generic subset of* Z .**

***(iv) The centre of* Z *has finite index in* Z *, and the points of* Z *whose centralizer has the same dimension as* Z *form a generic subset of* Z .**

Question 1. Is it sufficient that $Z°$ be central in Z ?

Lemme 4. ***Let* G *be an algebraic group,* g *be a generic point of* G *, and* H *be a definable connected subgroup of* G *; then every generic point of* g.H *is generic in* G *(even when* g *is not generic over the parameters of* H *).***

Corollary 5. ***In a connected algebraic group, every generic point* g *is generic in the center of its centralizer* Z(Z) *; otherwise said,* rg(g/z) = dim(Z(Z)) , rg(z) = dim(G) - dim(Z(Z)) *, and for* g *to be generic in* Z *it is enough that* Z° *be central in* Z .**

We conclude the section by a result that will be useful later.

Theorem 6. ***We consider, in a group* G *of finite Morley rank, a generic point* g *with centralizer* Z *; then tfcae :***
(i)* g *is generic in the coset* g.Z° *.
(ii)* Z° *is commutative, and the points of* g.Z° *whose centralizer have the same dimension as* Z *form a generic subset of it.
(In the algebraic case it is enough than* Z° *be commutative.)

We observe in the proof that g is generic in the centralizer of g' , and that g' is generic in the centralizer of g .

Exemple 9. How to make an exemple, if possible algebraic, where Z° is commutative but not central in Z ?

SECTION 3

A digression on the minimal group of the generic

In a group of finite Morley rank, every point is contained in a smallest definable subgroup.

In a simple algebraic group, the centralizer of a generic point is also its minimal subgroup.

Lemma 8. ***Consider a group* A *of finite Morley rank*, *which is the minimal group of one of its point* a *; then*, *if* g *is a point of* A° *generic over* a , A *is also the minimal group of* a.g .**

Remark 2. **An abelian torsion-free group abélien, of minimal dimension, whose generic (in fact every element !) is contained in a proper definable subgroup is a K-vector space of dimension two. According to Zil'ber, a field is definable in a torsion-free nilpotent group G ; indeed, in the quotient of G by its last center, each point belongs to the image of its centralizer modulo the last-but-one center.**

Corollaire 9. ***In a group of finite Morley rank, each generic point is generic in its minimal group.***

SECTİON 4

Generous centralizers

Assume that G acts on A . Let B be a definable subset of A , and N be its normalizer ; the set of conjugates of B can be identified with G/N ; for each integer r we note B_r the set of x's in B such that the conjugates of B going through x form a set of rank r .

Jaligot's Lemma. ***If*** **B_r** ***is not void,*** **$\dim(\cup_{g\in G} B_r^g) = \dim(G) - \dim(N) + \dim(B_r) - r$.**

Cherlin's proof. Consider the set C of (x,y) where x is in $\cup B_r^g$ and y is a conjugate of B to which x belongs.

Définition 1. **A definable subset of G is generous if the union of its conjugates is generic.**

Lemma 12. ***Consider, in a group* G *of finite RM,* H *a definable subgroup and an element a normalizing* H° *; the coset* aH *is generous iff :***

***(i)* H° *has finite index in the normalizer* N *of* aH**

(ii) the points of the coset which belongs only to a finite number of its conjugate form a generic subset of it.

Corollaire 13. ***A definable subgroup* H *of* G *is generous iff :***

(i)* H *has finite index in its normalizer

(ii) the points of* H *which belongs only to a finite number of its conjugate form a generic subset of it.

Remark that if A is the minimal subgroup of A , it is generous iff it has finite index in its normalizer.

Theorem 14. *If* **G** *has finite RM and* **a** *and* **g** *are two elements of* **G** , **g** *being generic over* **a** , *then* **g** *commutes with only a finite number of conjugates of* **a** .

Question 3. In an existentially closed group, if a ≠1 , every b commutes with an infinite number of conjugates of a . Is this property compatible with MC-condition, linearity, stability, superstability, or omega-stability ?

Lemma 15. *If* **G** *is connected,* **a** *has a generous centralizer, and* **g** *is generic over a , the connected component of the centralizer of* **a** *contains the connected component of the centralizer of a conjugate of* **g** .

In the two next lemmata, we consider the following situation : G is connected, and the centralizer Z of its generic point g is generous.

<u>Lemma 16.</u> ***The coset*** **g.*Z°*** ***is generous, and*** **g** ***is a generic point of it.***

<u>Lemma 17.</u> ***The coset*** **g.*Z°*** ***contains only a finite number of conjugates of*** **g .**

Theorem 18, and last. ***In a connected group of finite RM tfae :***
(ı) The generic centralizers are generous.
(ıı) A generic point g is generic in its centralızer **Z** ***, which contains only a finite number of conjugates of*** **g .**
(ııı) When g is generic, **z** ***and*** **c** ***are independent.***
(ıv) The generic centralizers are conjugate.
(v) The centers of the generic centralizers are conjugate.
(vı) **G** ***has an abelian generous subgroup.***
(vıı) **G** ***has a generous commutative subset (non nec. definable)***
(vııı) There exists a commutative coset, modulo a connected definable subgroup, which is generous.