Generic Multiple Transitivity in the Finite Morley Rank Setting

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Outline

1. Some Model Theory (Morley rank and genericity)
2. Some Group Theory (Multiple transitivity)
3. A Merging (Generic multiple transitivity)

Groups

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Examples

$\langle \mathbb{Z}, +, -, 0\rangle$, $\langle \mathbb{C}^*, \cdot, {}^{-1}, 1\rangle$ $\langle \mathbb{R}, +, -, \cdot, 0, 1\rangle$

Definability in a Group

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The quotient of a definable set with a definable relation is also definable.

Definability in a Field

A *definable subset* of a field

$$\langle F, +, -, \cdot, 0, 1 \rangle$$

is a subset of F^n which can be expressed by formulas of first order logic, i.e. formulas of finite length that contain only

- logical symbols $=, \wedge, \vee, \neg, \exists, \forall, (,)$
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- If we view $\mathbb{R}$ or $\mathbb{C}$ as an additive group, then there is no other definable subset.
- If we view $\mathbb{R}$ as a field, then

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  is definable. Therefore, all intervals are definable.
- (Tarski) If F is an algebraically closed field, then finite and co-finite subsets are the only definable subsets.

More Examples

In a group $\langle G, \cdot, ^{-1}, e \rangle$, the center of the group

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and conjugacy classes

$$\{g \in G \mid (\exists x)(g = x^{-1} \cdot a \cdot x)\}$$

are definable.

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Variables must be over elements, not over sets.
$\mathbb{R}$ is not definable in $\langle \mathbb{C}, +, -, \cdot, 0, 1 \rangle$.

Morley Rank

Let $\langle M, \ldots \rangle$ be a group or a field, $A \subseteq M^n$ a definable set. The *Morley rank* of A is defined as:

1. $\text{rk}(A) \geqslant 0$ iff A is nonempty.
2. $\text{rk}(A) \geqslant n + 1$ iff there exists a sequence of non-empty definable sets $\{A_i\}_{i=1}^{\infty}$ such that $A_i \subseteq A$, $A_i \cap A_j = \emptyset$ if $i \neq j$ and $\text{rk}(A_i) \geqslant n$.

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- $\text{rk}(\mathbb{C}, +, \cdot) = 1$, $\text{rk}(\mathbb{R}, +, \cdot) = \infty$.

Some Properties of Morley Rank

Observations

Let A and B be definable subsets in a structure of finite Morley rank.

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- $\mathrm{rk}(A \times B) = \mathrm{rk}(A) + \mathrm{rk}(B)$.
- If A is a group and B a normal subgroup, then $\mathrm{rk}(A/B) = \mathrm{rk}(A) - \mathrm{rk}(B)$.

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Conjecture. (Cherlin–Zilber)

Infinite simple groups of finite Morley rank are algebraic groups over algebraically closed fields.

Morley Degree

Definition

Let A be a non-empty set such that for every definable non-empty subset $B \subseteq A$, either $\mathrm{rk}(B) < \mathrm{rk}(A)$ or $\mathrm{rk}(A \setminus B) < \mathrm{rk}(A)$. Then we say A has (Morley) degree 1.

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If A is a disjoint union of d definable sets of degree 1, then we say A is of (Morley) degree d.

Rank and Degree Computations

Observation

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Corollary

A group of finite Morley rank satisfies the DCC on its definable subgroups.

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Proof.

At each step, either the degree or the rank decreases. □

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Theorem

A group is connected iff it has Morley degree 1.

Examples. $\mathrm{GL}_n(K)$ and $\mathrm{SL}_n(K)$ are connected.
$[\mathrm{O}_n(K) : \mathrm{SO}_n(K)] = 2$, so $\mathrm{O}_n(K)$ is not connected.

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PART II: Some Group Theory

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Definition

Let G act on X. If for every pairwise distinct $x_1, \ldots, x_n \in X$ and pairwise distinct $y_1, \ldots, y_n \in X$, there exists a (unique) $g \in G$ such that $gx_i = y_i$ for all $i = 1, \ldots, n$, then we say G acts (sharply) n-transitively on X.

Sharp Multiple Transitivity

Definition

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Examples

For all $n \geqslant 1$, S_n acts sharply n-transitively (also sharply $(n-1)$-transitively) on $\{1, \ldots, n\}$. For all $n \geqslant 3$, A_n acts sharply $(n-2)$-transitively on $\{1, \ldots, n\}$.

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Non-examples

K^* on K^+, $\mathrm{GL}_2(K)$ on K^2, where K is a field.

Sharp 2 or 3-transitivity

Example

For any field K, $K^* \ltimes K^+$ acts sharply 2-transitively on K, and $PGL_2(K)$ acts sharply 3-transitively on $\mathcal{P}_1(K)$.

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Sharply 2 or 3-transitive finite groups were classified by Zassenhaus in 1936.

The problem for infinite groups is still open.

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Theorem (Jordan, 1872)

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Higher Transitivity

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Theorem (Tits, 1952, and Hall, 1954)

There is no infinite group with a sharp n-transitive action, for $n \geqslant 4$.

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Let G be a reductive algebraic group acting algebraically on an irreducible variety V. For $n \geqslant 2$, if G acts n-transitively on V, then either $n = 2$, and the action is PGL_{m+1} on $\mathcal{P}_m$; or $n = 3$ and the action is PGL_2 on $\mathcal{P}_1$.

An Alternative Definition for n-transitivity

Observation

Let G be a group acting on a set X and $n \geqslant 2$. Then G acts n-transitively on X iff G acts transitively on $X^n \setminus E$, where $E = \{(x_1, \ldots, x_n) \mid x_i = x_j \text{ for some } i \neq j\}$.

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Note that E is 'small', hence $X^n \setminus E$ is 'large' or 'generic'.

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Note that E is 'small', hence $X^n \setminus E$ is 'large' or 'generic'. Therefore, we can relax the condition on $X^n \setminus E$ while keeping it large, and obtain new and natural examples.

Algebraic Actions

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If the induced action of G on V^n is transitive on an open subset, then Popov calles it a generically n-transitive action.

Theorem (Popov, 2007)

If characteristic is 0, among simple algebraic groups, only those of type A_n have generically 5-transitive or higher actions.

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PART III: Generic Multiple Transitivity

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Examples

All cofinite sets are generic in an infinite set. The set of linearly independent pairs of vectors is generic in $K^2 \times K^2$.

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Example

Let K be a field, then K^* acts generically sharply transitively on K^+, but not sharply transitively.

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- $\mathrm{PGL}_{n+1}(K)$ on $\mathcal{P}_n(K)$ is generically sharply $(n+2)$-transitive.

Question

Motivating Question (Borovik-Cherlin, 2008)

Let G be a connected group acting on a connected abelian group V definably, faithfully and generically sharply n-transitively. If $n = \mathrm{rk}(V)$, then is it true that V has a vector space structure of dimension n over an algebraically closed field K and $G \cong \mathrm{GL}_n(K)$?

A Possible Solution

Ongoing work with Alexandre Borovik.

Setting

Let G be a connected group acting on a connected abelian group V definably, faithfully and generically sharply n-transitively such that $n = \text{rk}(V)$ and V is not a 2-group.

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- V can be coordinatized, for example as $V = \bigoplus C_V^-(e_i)$, where $e_i = (1, \ldots, -1, \ldots, 1)$,

A Possible Solution

Ongoing work with Alexandre Borovik.

Setting

Let G be a connected group acting on a connected abelian group V definably, faithfully and generically sharply n-transitively such that $n = \text{rk}(V)$ and V is not a 2-group.

Then

- $\text{rk}(G) = n^2$,
- $S_n \ltimes (\mathbb{Z}_2)^n$ lies in G,
- V can be coordinatized, for example as $V = \oplus C_V^-(e_i)$, where $e_i = (1, \ldots, -1, \ldots, 1)$,
- and hence $V \cong F^n$ for some algebraically closed field F.

Hyperoctahedral Action

Theorem (in progress)

Let a group G act on a connected group V definably and faithfully. Assume V is an elementary abelian p-group (where $p \neq 2$) of Morley rank n, and $\mathrm{S}_n \ltimes (\mathbb{Z}_2)^n \leqslant G$. If G is infinite, then there exists an algebraically closed field F such that $V \cong F^n$, and G is isomorphic to one of the following:

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- a finite extension of $\mathrm{O}_n(F)$, or

Hyperoctahedral Action

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- an extension of $\mathrm{SL}_n(F)$ lying in $\mathrm{GL}_n(F)$, or
- a finite extension of $\mathrm{O}_n(F)$, or
- a finite extension of T, for some definable $T \leqslant (F^*)^n$,

and the action is the natural action in every case.

Answer

Note that $\mathrm{rk}(\mathrm{GL}_n) = n^2$, $\mathrm{rk}(\mathrm{SL}_n) = n^2 - 1$, $\mathrm{rk}(\mathrm{O}_n) = n(n-1)/2$, and $\mathrm{rk}(T) \leqslant n$.

Answer

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Corollary

Let G be a group acting on a connected abelian group V of Morley rank n. If V is not a 2-group and the action is definable, faithful and generically sharply n-transitive, then $G \cong \mathrm{GL}_n(F)$ and $V \cong F^n$ for some algebraically closed field F.

Teşekkürler

Thank you very much

Teşekkürler

Thank you very much and one more thing!

Other Meetings Involving Group Theory in Turkey

- Models and Groups Workshop II, Istanbul, 27-29 March 2014
- Antalya Algebra Days XVI, Antalya, 9-13 May 2014